4.1. Stein spaces

ly, let $\mathscr{A}$ be a sheaf of $\mathcal{O}_Y$-algebras, which is coherent as sheaf of $\mathcal{O}_Y$-modules. Then there exists an analytic space $(X, \mathcal{O}_X)$ and a finite morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f_* (\mathcal{O}_X)$ is isomorphic with $\mathscr{A}$ as sheaf of $\mathcal{O}_Y$-algebras ; the triple $(X, \mathcal{O}_X, f)$ is unique up to an isomorphism.

We do not prove this proposition here and refer to Houzel [6] or Narasimhan [9] for this proof. We note also that a proof of the direct part can be given along the same lines as theorem 3.1.3, combined with the fact that direct images under finite morphism preserve exact sequences of sheaves of $\mathcal{O}_X$-modules (in other words, that higher direct images are zero). We note also that, for *proper* morphisms (not necessarily finite), a much deeper result has been proved by Grauert [2], [3].

Finally, we remark that, in the real case, proposition 3.3.2. is false (take, for instance, X the submanifold of $\mathbf{R}^2$ defined by $x_2 - x_1^2 = 0$, $Y = \mathbf{R}$ and $f =$ the projection on the x_2-axis ; $f_* (\mathcal{O}_X)$ has support $x_2 \geqslant 0$, which is not an analytic subset of $\mathbf{R}$, hence $f_* (\mathcal{O}_X)$ cannot be coherent!)

Chapter 4.

THE FINITENESS THEOREM

In this chapter, we consider only *complex* analytic spaces, separated and having a countable basis of open sets.

4.1. *Stein spaces*

Let $(X, \mathcal{O}_X)$ be an analytic space, and K a subset of X ; we denote, as usual by $\hat{K}$ the set

$$\left\{ x \in X \mid \forall f \in \Gamma(X, \mathcal{O}_x) : |f(x)| \leqslant \sup_{y \in K} |f(y)| \right\}$$

Definition 4.1.1. a) $(X, \mathcal{O}_X)$ is called holomorphically convex if, for any K compact $\subset X$, $\hat{K}$ is compact ;
b) $(X, \mathcal{O}_X)$ is called a Stein space if it is holomorphically convex, and if, for any $x \in X$, there exist sections $f_1, ..., f_p \in \Gamma(X, \mathcal{O}_X)$ with $f_i(x) = 0$, such that x is an isolated point of the counter-image of 0 in the morphism $(X, \mathcal{O}_X) \to \mathbf{C}^p$ defined by $f_1, ..., f_p$, (This last property can also by expressed as the fact that the morphism of germs : $(X, \mathcal{O}_X, x) \to (\mathbf{C}^p, 0)$ defined by $f_1, ..., f_p$ is finite).

If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (" Theorems A and B " of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space $(X, \mathcal{O}_X)$. Then

1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
2) For $p \geqslant 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let $(X, \mathcal{O}_X)$ be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; if F is an analytic coherent sheaf on X, let $\tilde{F}$ be the trivial extension of F to U ; then $\tilde{F}$ is a coherent sheaf of $\mathcal{O}_U$ modules, and theorems A and B are valid for $\tilde{F}$: therefore, they are true for F.

4.2. *Topology on* $\Gamma(X, F)$.

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; and, with the previous notations, suppose that $\tilde{F}$ admits a *finite presentation* i.e. an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \to 0 .$$

Applying theorem B to the exact sequences

$$0 \to \operatorname{Im} \alpha \to \mathcal{O}_U^p \to \tilde{F} \to 0 \qquad \text{and} \qquad 0 \to \operatorname{Ker} \alpha \to \mathcal{O}_U^q \to \operatorname{Im} \alpha \to 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U,\alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U,\beta)} \Gamma(U, \tilde{F}) \to 0 .$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\operatorname{Im} \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\operatorname{Im} \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \operatorname{Im}(\alpha_x)$, hence $f \in \Gamma(U, \operatorname{Im} \alpha)$; but, according to theorem B, the mapping $\Gamma(U, \mathcal{O}_U)^q \to \Gamma(U, \operatorname{Im} \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \operatorname{Im} \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of $\tilde{F}$ (in fact, it does not even depend on the imbedding $X \to U$, but we shall not need it here). For, suppose we have a second presentation