4.4. The finiteness theorem

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Remark. If X is not separated, an intersection of two open Stein subspaces of X need not be Stein; take f.i. for X two copies of C^2, identified everywhere except at O; there is an obvious covering of X by two open subspaces, identicals with C^2; but their intersection is $C^2 - \{o\}$, and therefore is not Stein!

4.4. *The finiteness theorem*

Theorem 4.4.1. (Cartan — Serre). Let X be a compact analytic space, and F be a coherent analytic sheaf on X. Then, for every $p \geqslant 0$ $H^p(X, F)$ is separated and finite dimensional.

We shall give two proofs of this theorem ; both are interesting for further applications.

1st proof. Let $\{X_i\}$ and $\{X_i'\}$ be two finite coverings of X of the type considered in the previous articles, such that, for every i, X'_i is relatively compact in X_i. Then, if we denote by $\mathcal{U}$ (resp. $\mathcal{U}'$) the covering $\{X_i\}$ (resp. $\{X_i'\}$), the natural restriction map $C^p(\mathcal{U}, F) \to C^p(\mathcal{U}', F)$ is compact.

Consider now the map

$$(\rho, d) : Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \to Z^p(\mathcal{U}', F)$$

this map is surjective, and we have $(, d\rho) = (\rho, 0) + (0, d)$, $(\rho, 0)$ being compact ; then the following lemma proves that Im $(0, d)$ is closed and finite codimensional, q.e.d.

Lemma 4.4.2. Let E and F two Frechet spaces, u_1 and u_2 two linear continuous maps $E \to F$ such that $u_1 + u_2$ is surjective, and u_1 compact. Then Im (u_2) is closed and finite codimensional. For the proof, see e.g. [5].

2nd proof. Consider $\mathcal{U}$ and $\mathcal{U}'$ as above, and consider the map
(ρ, d) $C^{p-1}(\mathcal{U}, F)/Z^{p-1}(\mathcal{U}, F) \to [C^{p-1}(\mathcal{U}', F)/Z^{p-1}(\mathcal{U}', F)] \oplus Z^p(\mathcal{U}, F)$
(ρ, d) is clearly injective. I claim that its image is closed: In fact, since $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is injective, this image consists of the pairs $(\bar{a}', b)$, $a' \in C^{p-1}(\mathcal{U}', F)$, $b \in Z^p(\mathcal{U}, F)$ such that $da' = \rho\, b$, which proves the assertion.

Now we have $(\rho, d) = (\rho, 0) + (0, d)$ and $(\rho, 0)$ is compact. By a well-known lemma, it results that Im $(0, d)$ is closed, which means that $H^p(\mathcal{U}, F)$ is separated.

Finally, since $\bar{\rho}$ is compact, and is an isomorphism, it follows that the identity map of $H^p(\mathcal{U}, F)$ into itself is compact ; therefore this space is finite dimensional ; this proves the theorem.