**Zeitschrift:** L'Enseignement Mathématique
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique
**Band:** 14 (1968)
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** FLATNESS AND PRIVILEGE

**Autor:** Douady, A.

**DOI:** https://doi.org/10.5169/seals-42343

# FLATNESS AND PRIVILEGE

by A. DOUADY

## I. FLAT MORPHISMS

### § 1. *Analytic subspaces of an analytic space*

Let $Y_1$ and $Y_2$ be closed analytic subspaces of an analytic space $X$, and let them be defined by the $\mathcal{O}_X$ ideals $J_1$, $J_2$.

*Definition 1:* We say that $Y_1$ is *analytically included* in $Y_2$, and we write $Y_1 \subset Y_2$, when $J_1 \supset J_2$.

*Remark:* The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example: $X = (\mathbf{C}, \mathcal{O}_\mathbf{C})$; $J_1 = (x)$, $J_2 = (x^2)$. The space $Y_1$ is a simple point, $Y_2$ is a double point, $Y_1 \not\supset Y_2$, while they have the same underlying set.

*Definition 2:* The subspace $Y_1 \cup Y_2$ is the smallest subspace of $X$ containing $Y_1$ and $Y_2$, and it is defined by $J_1 \cap J_2$. The subspace $Y_1 \cap Y_2$ is the biggest subspace of $X$ contained in both $Y_1$ and $Y_2$, and it is defined by $J_1 + J_2$.

*Remark:* The underlying set of $Y_1 \cup Y_2$ (Resp. $Y_1 \cap Y_2$) is the union (Resp. intersection) of the underlying sets of $Y_1$ and $Y_2$. However $\cup$ and $\cap$ of analytic spaces do not satisfy the distributivity laws which hold in set-theory: $(Y_1 \cup Y_2) \cap Y_3$ contains $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$, and therefore their union; similarly $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$. In general the converse inclusions do not hold.

Example: Let $X = \mathbf{C}^2$ and $Y_1$, $Y_2$, $Z$ be given by ideals $(x-y)$, $(x+y)$ and $(x)$ respectively.

$(Y_1 \cup Y_2) \cap Z$ is $\{0\}$ provided with $\mathbf{C}\{y\}/(y^2)$, while $(Y_1 \cap Z) \cup (Y_2 \cap Z)$ is the reduced space $\{0\}$. On the other hand: $Y_1 \cap Y_2 \subset Z$, $(Y_1 \cap Y_2) \cup Z = Z$, while $(Y_1 \cup Z) \cap (Y_2 \cup Z)$ is the space defined by the ideal $(x^2, xy)$. Its local ring at the origin is $\mathbf{C}\{x, y\}/(x^2, xy)$ in which $x$ is nilpotent.

*Definition 3:* Let $X'$, $X$ be analytic spaces, $Y$ a closed analytic subspace of $X$ defined by $J$, and $f = (f_0, f^1) : X' \to X$ a morphism.

*The inverse image* of $Y$ by $f$, $f^{-1}(Y)$, is the analytic subspace $Y'$ of $X'$ defined by the ideal $J' = f^1(J)\,\mathcal{O}_{X'}$.

The inverse image of a simple point $x$ in $X$ is called the $f$-fiber over $x$, and is denoted by $f^{-1}(x)$ or $X'(x)$.

*Proposition 1:* If $f = (f_0, f^1) : X' \to X$ is a morphism of analytic spaces, and $Y$ is a subspace of $X$, then $f^{-1}(Y) \simeq Y \underset{X}{\times} X'$.

*Proof:* Let $T$ be any analytic space, and $g : T \to X'$ a morphism. Then $g$ can be considered as a morphism from $T$ to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from $T$ to $Y$. Thus $f^{-1}(Y)$ and $X' \underset{X}{\times} X$ are solutions of the same universal problem.

## § 2. *Analytic pull-back*

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \underset{A}{\otimes} F$, where $A$ is a commutative ring and $E$, $F$ are two $A$-modules.

(1°) $E \otimes A^n = E^n$ $(n \in N)$

(2°) If the sequence of $A$-modules $F' \to F \to F'' \to 0$ is exact, then also the sequence $E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}$; $f_{ij} : F_j \to F_i$ is an inductive system, then

$$E \otimes \underset{\to}{lim}\, F_i = \underset{\to}{lim}\,(E \otimes F_i).$$

On the other hand these properties characterize completely the functor $\otimes$.

*Definition 1:* Let $f = (f_0\ f^1) : X' \to X$ be a morphism of analytic spaces, and $\mathcal{E}$ an $\mathcal{O}_X$-module. Then $f_0^* \mathcal{E}$ is an $f_0^*\,\mathcal{O}_X$-module and $\mathcal{O}_{X'}$ is also an $f_0^*\,\mathcal{O}_X$-module (by $f^1 : f_0^*\,\mathcal{O}_X \to \mathcal{O}_{X'}$).

*The analytic pull-back* $f^*\mathcal{E}$ of $\mathcal{E}$ by $f$ is defined by scalar extension:

$$f^*\mathcal{E} = f_0^*\mathcal{E} \underset{f_0^*\mathcal{O}_X}{\otimes} \mathcal{O}_{X'}$$

*Remark:* The inverse image is a particular case of the analytic pull-back.

In fact, if $Y$ is a closed analytic subspace of $X$ and $f : X' \to X$ is a morphism:

$$f^* \mathcal{O}_Y = f_0^* (\mathcal{O}_X / J_Y) \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'} / f^1 (J_Y) . \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that $A/I \underset{A}{\otimes} E \simeq E/IE$).

*Elementary properties of the analytic pull-back:*

(*a*) $(f^* \mathscr{E})_{x'} = (f_0^* \mathscr{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathscr{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ where $x = f_0 (x')$ (since $\otimes$ commutes with inductive limits).

(*b*) $f^* (\mathscr{E} \underset{\mathcal{O}X}{\otimes} \mathscr{F}) = f^* \mathscr{E} \underset{\mathcal{O}X'}{\otimes} f^* \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are $\mathcal{O}_X$-modules.

(*c*) If $\mathscr{E}$ is a coherent $\mathcal{O}_X$-module, then $f^* \mathscr{E}$ is a coherent $\mathcal{O}_{X'}$-module.

In fact, $\mathscr{E}$ has a locally finite presentation: $\mathcal{O}_X^q \to \mathcal{O}_X^p \to \mathscr{E} \to 0$, and $f^*$ is compatible with cokernels, $f^* (\mathcal{O}_X^r) = \mathcal{O}_{X'}^r$.

*Special case:* The pull-back of vector bundle. Let $(E, \pi)$ be an analytic vector bundle over the analytic space $X$, and $f: X' \to X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over $X'$, such that $\bar{f}$ is a bundle morphism. We call this bundle $E'$.

$$\begin{array}{ccc} E \underset{X}{\times} X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

*Proposition 1:* Let $\mathscr{E}$ (Resp. $\mathscr{E}'$) be the sheaf of analytic sections of $E$ (Resp. $E'$). Then $\mathscr{E}' = f^* \mathscr{E}$.

*Proof (Sketch):* We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathscr{E} \to \mathscr{E}'$, which extends to a morphism $f^* \mathscr{E} \to \mathscr{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to $X'$, we can suppose that $E$ is a trivial bundle over $X$ with fiber $\mathbf{C}^r$, then $\mathscr{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathscr{E} = \mathscr{E}'$.

## § 3. *Introduction to flatness by examples*

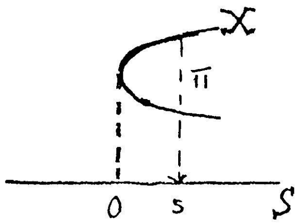
Let $S$ be an analytic space. By analytic space over $s$ we mean an analytic space $X$ provided with a morphism $\pi : X \to S$. Let $S$ be a simple point in $S$, and consider $X(s) = f^{-1}(s)$.

The main purpose of these lectures is to give a precise meaning to the expression:

" $X(s)$ depends nicely on $s$", and to give a criterion for the " nice " behaviour.

We begin with some examples.

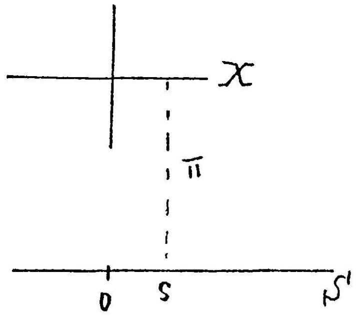
*Example 1 :* $X$ is the closed subspace on $\mathbf{C}^2$ defined by $(y^2-x)$, $S = \mathbf{C}$ and $\pi$ = 1st projection.




$$X(s) = \begin{cases} \text{2 simple points if } s \neq 0 \\ \text{double point if } \quad s = 0\,. \end{cases}$$

Here the behaviour of $X(s)$ is nice.

*Example 2 :* $X$ is the closed subspace of $\mathbf{C}^2$ defined by $(xy)$, $S = \mathbf{C}$ and $\pi$ = 1st projection.




$X(s)$ is given by $(x-s, xy)$, and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if} \quad s \neq 0 \\ \quad (x) & \text{if} \quad s = 0\,. \end{cases}$$

The first case is a simple point, the second one the, $y$-axis.

A similar example is the map of a point into $\mathbf{C}$.

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreductible component of $X$, and after removing this component $\pi$ behaves nicely.

This kind of removing is not possible in general, as the following example shows:

*Example 3 :* $X$ is given in $\mathbf{C}^3$ by $(xz-y)$, and $\pi$ is the projection on the $(x, \mathrm{y})$-plane.

If $s = (x_0, y_0)$, then the fiber $X(s)$ is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\dfrac{y_0}{x_0}\right) \text{ if } x_0 \neq 0 \\ (x, y) \qquad\qquad\qquad \text{if} \quad x_0 = y_0 = 0 \\ (1) \quad \text{if} \quad x_0 = 0\; y_0 \neq 0\,. \end{cases}$$

The set of "nice" fibers is dense in $X$, so we cannot remove the $z$-axis and still get a closed subspace of $\mathbf{C}_3$.

## § 4. *Algebraic study of flatness*

In the following all rings are commutative, with 1, and all modules are unitary.

*Definition 1:* An $A$-module $E$ is *flat*, if for every exact sequence of $A$-modules

$$0 \to F' \to F \to F'' \to 0 ,$$

the sequence $0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is also exact. We can also say, because $\otimes$ is right exact, that $E$ is flat, if for every injective homomorphism $F' \to F$, $E \otimes F' \to E \otimes F$ is also injective.

*Examples of modules which are not flat:*

(1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\,\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \to \mathbf{Z} \xrightarrow{2I} \mathbf{Z}$ $(2I : x' \to 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So $\mathbf{Z}_2$ is not a flat $\mathbf{Z}$ module.

(2) If $A = \mathbf{C}\{x\}$, $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$, $F = F' = \mathbf{C}\{x\}$, then the sequence $0 \to F \xrightarrow{xI} F'$ $(xI : p(x) \to xp(x))$ is exact. But the homomorphism $E \xrightarrow{xI} E$ is not injective.

*Proposition 1:* If $A$ is an integral domain and $E$ a flat $A$-module, then $E$ is torsion-free.

*Proof:* Let $a \in A$, $a \neq 0$. Because $A$ is an integral domain, the sequence $0 \to A A \xrightarrow{aI}$ $(aI : x \to ax)$ is exact. Since $E$ is flat, the sequence $0 \to E \xrightarrow{aI} E$ is also exact. In other words $E$ has no torsion elements.

*Proposition 2:* If $A$ is a principal-ideal domain, then $E$ is flat if and only if $E$ is torsionfree.

*Proof:* See corollary of prop. 6.

*Examples of flat modules:*

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if $E$ is free and finite type, then $E = A^n$ and $E \otimes F = F^n$. If $F' \to F$ is injective, so is $F'^n \to F^n$ too.

If $E$ is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of $E$ follows from (1).

(3) Let $S$ be a multiplicative system in $A$. Then the ring of fractions $S^{-1} A$ is a flat $A$-module. In fact the ring $S^{-1} A$ can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that $S$ has only regular elements. We can define in the set $S$ a partial order in the following way:

$$s' \geqq s \Leftrightarrow \exists\, t \in A\,, \quad ts = s' \quad \text{(such a } t \text{ is then unique).}$$

Let $E_s = A$ for every $s \in S$, and if $s' \geqq s$ (i.e. $s' = ts$) then let $f_s^{s'}$ be the homomorphism $t.\ I_A : E_s \to E_{s'}$. The family $(E_s)_{s \varepsilon S}$ with the homomorphisms $(f_s^{s'})$ is an inductive system.

Let $E = \varinjlim E_s$ be the inductive limit of this system, and $\varphi_s$ the canonical homomorphism $E_s \to E$. We shall define an isomorphism $\psi : E \to S^{-1} A$.

We first define for every $s$ a homomorphism $\psi_s : E_s = A \to S^{-1} A$; $x \to x/s$. Now if $s' \geqq s$, then

$$(\psi_s{}' \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x)\,.$$

Therefore there exists a homomorphism $\psi : E \to S^{-1} A$, satisfying $\psi_s = \psi \circ \varphi_s$ for every $s \in S$.

Because every element of $S^{-1} A$ has the form $a/s$, $\psi$ is surjective. On the other hand if $\psi\,(\phi_s(x)) = 0$, then $\psi_s(x) = x/s = 0$. Thus $x = 0$, and $\psi$ is also injective.

The above proof can be extended to the general case, not assuming that the elements of $S$ are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

*Theorem 1:* (Daniel, Lazard)

Any flat module is a inductive limit of free modules.

For the proof: See *C.R. Acad. Sci. Paris,* 258 (1964), pp. 6313-6316.

*Some elementary properties of flat modules:*

(1) If $E$ and $F$ are flat $A$-modules, then $E \underset{A}{\otimes} F$ is also flat. In fact, if $G' \to G$ is injective, then $F \otimes G' \to F \otimes G$ is injective, and also $E \otimes (F \otimes G') \to$ $\to E \otimes (F \otimes G)$ is injective. The result follows from the assosiativity of the tensor product.

(2) Let $\phi : A \to B$ be a ring homomorphism, and $E$ a flat $A$-module. The module $B \underset{A}{\otimes} E$ is a flat $B$-module.

If $F$ is a $B$-module, then $F \underset{B}{\otimes} (B \underset{A}{\otimes} E) = (F \underset{B}{\otimes} B) \underset{A}{\otimes} E = F \underset{A}{\otimes} E$ further if $F'$ and $F$ are $B$-modules, and $F' \to F$ an injective homomorphism of $B$-modules, we can consider this homomorphism as an injective homomorphism of $A$-modules. Because $E$ is $A$-flat,

$$F' \otimes_A E \to F \otimes_A E \text{ is injective.}$$

(3) Let $\phi : A \to B$ be a ring homomorphism, such that $B$ is a flat $A$-module. If $F$ is a flat $B$-module, then $F$ is a flat $A$-module. In fact: if $E' \to E$ is injective, then $E' \underset{A}{\otimes} B \to E \underset{B}{\otimes} B$ is injective, and also $(E \underset{A}{\otimes} B) \underset{B}{\otimes} F' \to (E \underset{A}{\otimes} B) \underset{B}{\otimes} F$ is injective. But $(E' \otimes_A B)_B \otimes F' = E' \otimes_A F$; $(E \otimes_A B) \otimes_B F = E \otimes_A F$.

If an $A$-module $E$ is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor.

*Definition 2: A free resolution* of $E$ is an exact sequence: $\ldots \to L_n \to L_{n-1} \to$ $\to \ldots \to L_1 \to L_0 \to E \to 0$, where all $L_i$ are free $A$-modules.

The complex of the resolution is the sequence

$$(\mathrm{L.}) \ \ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to 0 .$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensorproducts $L_i \otimes F$, we get

$$(\mathrm{L.} \otimes F) \ \ldots \to L_n \otimes F \to L_{n-1} \otimes F \to \ldots \to L_1 \otimes F \to L_0 \otimes F \to 0 .$$

*Definition 3:*

$$\mathrm{Tor}_n^A(E, \mathrm{F}) = H_n(\mathrm{L.} \otimes F) = \frac{\mathrm{Ker}(L_n \otimes F \to L_{n-1} \otimes F)}{\mathrm{Im}\,(L_{n+1} \otimes F \to L_n \otimes F)}$$

if $n \geqq 1$, and $\mathrm{Tor}_0^A(E, F) = \mathrm{Coker}\,(L_1 \otimes F \to L_0 \otimes F) = E \otimes F.$

*Basic properties of Tor:*

(1) $\mathrm{Tor}_n(E, F)$ is independent of the choice of the resolution (up to a canonical isomorphism).

(2) If we take a free resolution of $F$, we get $\mathrm{Tor}_n(F, E) = \mathrm{Tor}_n(E, F)$ (Symmetry of the Tor). We can also define $\mathrm{Tor}_n(E, F)$ by taking two free resolutions, one of $E$ and one of $F$.

(3) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence, then we get a long exact sequence:

$$\begin{array}{l} \mathrm{Tor}_n(E', F) \to \mathrm{Tor}_n(E, F) \to \mathrm{Tor}_n(E'', F) \to \\ \to \mathrm{Tor}_{n-1}(E', F) \to \mathrm{Tor}_{n-1}(E, F) \to \mathrm{Tor}_{n-1}(E'', F) \to \\ \to \text{— — — — — — — — — — — — — — — — —} \to \\ \to \mathrm{Tor}_1(E', F) \to \mathrm{Tor}_1(E, F) \to \mathrm{Tor}_1(E'', F) \to \\ \to E' \otimes F \to E \otimes F \to E'' \otimes F \to 0 . \end{array}$$

(4) Tor is compatible with inductive limit, i.e. if $E = \varinjlim (E_i)$, then

$\mathrm{Tor}_n(\varinjlim E_i, F) = \varinjlim (\mathrm{Tor}_n(E_i, F))$.

(5) We can define $\mathrm{Tor}_n(E, F)$ by taking a flat resolution of $E$.

*Proposition 3:* Let $E$ be an $A$-module. Then the following conditions are equivalent:

($a$) $E$ is flat.

($b$) For all $A$-modules $F$, and for all $n \geqq 1$, $\mathrm{Tor}_n(E, F) = 0$.

($c$) For all $A$-modules $F$, $\mathrm{Tor}_1(E, F) = 0$.

*Proof:* ($a$) $\Rightarrow$ ($b$). If $\ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to F \to 0$ is a free resolution of $F$, then the sequence

$$\ldots \to E \otimes L_n \to E \otimes L_{n-1} \to \ldots \to E \otimes L_1 \to E \otimes L_0 \to E \otimes F \to 0$$

is exact, thus $\mathrm{Tor}_n(E, F) = 0$ for all $n \geqq 1$.

($b$) $\Rightarrow$ ($c$) clear. ($c$) $\Rightarrow$ ($a$): If the sequence $0 \to F' \to F \to F'' \to 0$ is exact, so is also (by (3) above) $\mathrm{Tor}_1(E, F'') \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$. Now $\mathrm{Tor}_1(E, F'') = 0$, thus $E$ is flat.

*Proposition 4:* If $I$ and $J$ are two ideals in $A$, then $\mathrm{Tor}_1^A(A/I, A/J) = I \cap J / I . J$.

*Proof:* From the exact sequence $0 \to I \to A \to A/I \to 0$, we get the exact sequence:

$$\mathrm{Tor}_1(A, A/J) \to \mathrm{Tor}_1(A/I, A/J) \to I \otimes A/J \to A \otimes A/J \to A/I \otimes A/J \to 0 .$$

But now $\mathrm{Tor}_1(A, A/J) = 0$ ($A$ beeing $A$-free), and $I \otimes A/J = I/I . J$; $A \otimes A/J = A/J$. Therefore the sequence $0 \to \mathrm{Tor}_1(A/I, A/J) \to I/I . J \to A/J$ is exact, and $\mathrm{Tor}_1(A/I, A/J) = \mathrm{Ker}(I/I . J \to A/J) = I \cap J / I . J$.

*Example :* Let $U$ be an open set in $\mathbf{C}^n$, and $x \in U$. Further let $X, Y \subset U$ be two hypersurfaces, defined by $I = (f)$ and $J = (g)$. Supposing that $f$ and $g$ do not have common factors: $I_x \cap J_x = I_x J_x$, and

$$\operatorname{Tor}_1 (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = \operatorname{Tor}_1 (\mathcal{O}_{U,x}/I_x, \quad \mathcal{O}_{U,x}/J_x) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0 .$$

*Heuristic remark :* The formula $\operatorname{Tor}_1 (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ expresses the fact that $X$ and $Y$ are " in general position ". If for example $X$ and $Y$ are two linears subspaces in $\mathbf{C}^n$ of dimensions $p$ and $q$, we have $\operatorname{Tor}_1 (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ if $\dim (X \cap Y) = p+q-n$, and $\operatorname{Tor} (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$ otherwise.

Next we shall prove an elementary flatness criterion.

*Proposition 5 :* Let $E$ be an $A$-module. The following conditions are equivalent:

($a$) $E$ is flat.

($b$) For all finitely generated ideals $I$ of $A$, $\operatorname{Tor}_1 (E, A/I) = 0$.

($c$) For all monogenous $A$-modules $F$, $\operatorname{Tor}_1 (E, F) = 0$.

*Proof:* ($a$) $\Rightarrow$ ($b$), by prop. 3.

($b$) $\Rightarrow$ ($c$): Because Tor is compatible with inductive limit, we can suppose, that $\operatorname{Tor}_1 (E, A/I) = 0$ for an arbitrary ideal $I$ of $A$. But every monogenous $A$-module $F$ can be represented by $A/I$.

($c$) $\Rightarrow$ ($a$). By prop. 3 it is sufficient to prove that $\operatorname{Tor}_1 (E, F) = 0$ for any $A$-module $F$.

First consider the case, where $F$ is finitely generated. We use induction, supposing that $\operatorname{Tor}_1 (E, F) = 0$, when $F$ has $n$ generators. Let $F$ have $(n+1)$ generators $x_1, ..., x_n, x_{n+1}$. If $F'$ is the submodule generated by $\{x_1, ..., x_n\}$, then $F' \subset F$ and $F'' = F/F'$ is monogenous. The exact sequence $0 \to F' \to F \to F'' \to 0$ gives the exact sequence $\operatorname{Tor}_1 (E, F') \to \operatorname{Tor}_1 (E, F) \to \operatorname{Tor}_1 (E, F'')$. Now $\operatorname{Tor}_1 (E, F') = \operatorname{Tor}_1 (E, F'') = 0$, thus $\operatorname{Tor}_1 (E, F) = 0$. In the general case, $F$ can be considered as an inductive limit of finitely generated modules, and because Tor is compatible with inductive limits, $\operatorname{Tor}_1 (E, F) = 0$.

*Proposition 6 :* Let $A$ be an integral domain, and $E$ an $A$-module. Then $E$ is torsionfree if and only if $\operatorname{Tor}_1 (E, A/(a)) = 0$, for any element $a \in A$.

*Proof:* If $E$ is $A$-module, $a \in A$, then the exact sequence $0 \to A \xrightarrow{aI} A \to A/(a) \to 0$ gives the exact sequence $0 \to \operatorname{Tor}_1 (E, A/(a)) \to E \xrightarrow{aI} E$. In other words $\operatorname{Tor}_1 (E, A/(a)) = \{x \in E \mid ax = 0\}$, from which the result follows.

*Corollary:* Let $A$ be a principal ideal domain. $E$ is flat if and only if $E$ is torsionfree.

*Proof:* We have already proved that, if $E$ is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

*Theorem 2:* Let $A$ be a noetherian local ring with maximal ideal $m$; $k = A/m$, and $E$ a finitely generated $A$-module. The following conditions are equivalent:

(*a*) $E$ is free.

(*b*) $E$ is flat.

(*c*) $\mathrm{Tor}_1^A(E, k) = 0$.

*Proof:* We have already proved $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$: We recall first Nakayma's lemma. If $A$ is a local ring with maximal ideal $m$; $k{=}A/m$, and $E$ is a finitely generated $A$-module, such that $k \underset{A}{\otimes} E = E/mE = 0$, then $E = 0$.

The module $\bar{E} = k \underset{A}{\otimes} E = E/mE$ is a finitely generated vector space over $k$. Let $\{\bar{x}_1. \ldots, \bar{x}_r\}$ be a base of $\bar{E}$ (over $k$), and $\{x_1, \ldots, x_r\}$ $E$ representatives of $\bar{x}_i : s$. Consider the homomorphism $\phi : A^r \to E$, $\phi(a_1, \ldots, a_r) = {}$ $= \sum a_i x_i$. Denoting by $R$ and $Q$ the kernel and the cokernel of $\phi$, we get an exact sequence:

$$(*) \qquad 0 \to R \to A^r \to E \to Q \to 0$$

and $R$, $Q$ are finitely generated $A$-modules. From (*) we get the exact sequence

$$A^r \underset{A}{\otimes} k \to E \underset{A}{\otimes} k \to Q \underset{A}{\otimes} k \to 0\,.$$

But $\bar{E} = E \underset{A}{\otimes} k \simeq k^r = A^r \underset{A}{\otimes} k$, so $Q \underset{A}{\otimes} k = 0$, and by Nakayama's lemma $Q = 0$.

Therefore ge have an exact sequence

$$0 \to R \to A^r \to E \to 0\,.$$

From this we get: $\mathrm{Tor}_1(E, k) \to k \underset{A}{\otimes} R \to k^r \to \bar{E} \to 0$ (exact). Now: $\bar{E} \simeq k^r$, $\mathrm{Tor}_1(E, k) = 0$ (by assumption). Therefore $k \underset{A}{\otimes} R = 0$, and once more by Nakayama's lemma $R = 0$, thus $E \simeq A^r$, i.e. $E$ is free.

*Proposition 7:* Let $\phi : A \to B$ be a ring homomorphism, and let B be $A$-flat. If $I$ is an ideal of $A$, we write $\bar{A} = A/I$, $\bar{B} = B/IB = \bar{A} \otimes_A B$. Let $F$ be a $B$-module, then: $\mathrm{Tor}_i^A(\bar{A}, F) = \mathrm{Tor}_i^B(\bar{B}, F)$ $(i \geqq 0)$.

*Proof:* We choose first a $B$-free resolution of $F$

$$\to L_{n+1} \to L_n \to \ldots \to L_1 \to L_0 \to F \to 0 \,.$$

If $L.$ is the respective complex of resolution, then

$$\bar{B} \otimes_B L. = B/IB \otimes_B L. = \bar{A} \otimes_A (B \otimes_B L.) = \bar{A} \otimes_A L.$$

Because every $L_i$ is $B$-free, and $B$ is $A$-flat, every $L_i$ is $A$-flat (Property 3 after Th. 1). Thus $L.$ is a flat $A$-resolution, and

$$\mathrm{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \otimes_A L.) = H_i(\bar{B} \otimes_B L.) = \mathrm{Tor}_i^B(\bar{B}, F) \,.$$

We shall next state the second flatness criterion for noetherian local rings.

*Theorem 3:* Let $A$ and $B$ be two noetherian local rings, with maximal ideals $\underline{m}$, $\underline{n}$; $k = A/\underline{m}$. If $\phi : A \to B$ is a local homomorphism (i.e. $\phi(\underline{m}) \subset \underline{n}$), and $F$ finitely generated $B$ module then

$$F \text{ is A-flat} \Leftrightarrow \mathrm{Tor}_1^A(k, F) = 0 \,.$$

The proof of this theorem is much more difficult than that of th. 20 see for example:
Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, $(i) \Leftrightarrow (iii)$, p. 98.

The conditions in Bourbaki's theorem are here fullfilled:

1° A finitely generated module $F$ over a noetherian local ring $B$ is idealwise separated for $\underline{n}$. (*Ibid.*, § 5. 1. Ex. 1, p. 97.)

2° If $\phi : A \to B$ is a local homomorphism, $F$ is also idealwise separated for $\underline{m}$. (*Ibid.*, § 5, prop. 2, p. 101.)

3° Also the flatness condition is fulfilled, because $k$ is a field.

*Remark:* The main interest of the theorem lies in the fact, that it is true without any assumption of finitness on $B$.

*Corollary:* If the assumptions are the same as in the theorem 3, and if moreover $B$ is $A$-flat, then

$$F \text{ is } A\text{-flat} \Leftrightarrow \mathrm{Tor}_1^B(\bar{B}, F) = o \,,$$

where $\bar{B} = B/mB$.

*Proof:* $\mathrm{Tor}_1^A(k, F) = \mathrm{Tor}_1^B(\bar{B}, F)$, by prop. 7.

## § 5. *Geometric applications of the flatness criterions*

### A) *Flatness for finite morphisms*

*Proposition 1:* Let $\pi\colon X \to S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_*(\mathcal{O}_X)$ is a coherent analytic sheaf over $S$. The following conditions are equivalent:

($a$) $\pi$ is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, $s = \pi(x)$).

($b$) For every $s$, $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$-module.

($c$) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

*Proof:* Because $\pi$ is finite $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \varepsilon \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is ($b$) $\Rightarrow$ ($c$).

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

*Proposition 2:* Let $S$ be a reduced analytic space and $\mathcal{E}$ a coherent $\mathcal{O}_s$-module. Let $E(s)$ be the finite dimensional vector space (over $\mathbf{C}$) $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s$. $\mathcal{E}$ is a locally free $\mathcal{O}_{S,s}$-module if an only if $\dim_{\mathbf{C}} E(s)$ is locally constant.

*Proof:* If $\mathcal{E}$ is locally free, then $\dim_{\mathbf{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbf{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \to \mathcal{E}_U \to 0$ is exact. $d$ is determined by a $p \times q$ matrix of analytic functions on $U$, so it gives a morphism $\mathbf{C}_U^p \xrightarrow{d} \mathbf{C}_U^q$ of trivial vector bundles over $U$.

From the exact sequence $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \to \mathcal{E}_s \to 0$, we get (by making tensor-products with $\mathbf{C}_s$) the exact sequence:

$$\mathbf{C}_s^p \xrightarrow{d(s)} \mathbf{C}_s^q \to E(s) \to 0\,,$$

which shows that $d$ has constant rank in $U$. Thus Ker $d$ and Im $d$ are vector bundles, and we can write

$$\mathbf{C}_U^p = F_1 \oplus G_1\,, \qquad \mathbf{C}_U^q = F_0 \oplus G_0\,,$$

$$d : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_0\,. \end{cases}$$

Now $\mathscr{E} \simeq$ the sheaf of analytic sections of $G_0$, therefore $\mathscr{E}$ is locally free.

*Definition 1:* Let $\pi : X \to S$ be a finite morphism of analytic spaces, and $s \in S$. For each $x \in X(s) = \pi^{-1}(s)$, $\mathcal{O}_{X(s),x} = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is finite dimensional vectorspace over $\mathbf{C}$. Denote its dimension by $\nu(x)$. Then the degree $\nu(s)$ of $s$ is defined by $\nu(s) = \sum_{x \varepsilon X(s)} \nu(x)$.

*Theorem 1:* Let $\pi : X \to S$ be a finite morphism of analytic space and let $S$ be a reduced space. Then $X$ is flat over $S$ if and only if $\nu(s)$ is locally constant function of $s$.

*Proof:*

$$\nu(s) = \sum_{x \varepsilon X(s)} \dim_{\mathbf{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbf{C}} \Big( \bigoplus_{x \varepsilon X(s)} \mathcal{O}_{X(s),x} \Big)$$

$$= \dim_{\mathbf{C}} \Big( \bigoplus_{x \varepsilon X(s)} (\mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}) \Big)$$

$$= \dim_{\mathbf{C}} \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \pi_* (\mathcal{O}_X)_s = \dim_{\mathbf{C}} E(s) .$$

The theorem follows from propositions 1 and 2.

*Examples of flat morphisms*

*Example 1:* If $\pi : X \to S$ is a local isomorphism near $x$, then $\pi$ is flat at $x$.

*Example 2:* Consider § 2, Ex. 1. Here $\nu(x) = 1$.

*Examples of non-flat morphisms*

*Examples 1:* If $X \subset S$ is a closed subspace, not open, $\nu(s)$ is not locally constant.

*Example 2:* Let $X$ be a subspace of $\mathbf{C}^4$ defined by the ideal intersection of $(x_3, x_4)$ and $(x_1 - x_1, x_4 - x_2)$ (which is equal to the product ideal) and let $\pi$ be the projection onto the $(x_1, x_2)$-plane $\mathbf{C}^2$. Then $X$ is a union of two 2-planes in $\mathbf{C}^4$, whose intersection is (0). When $s \neq 0$. $X(s)$ consists of two simple points, so $\nu(s) = 2$. $X(0)$ is given by the ideal $(x_1, x_2, x_3^2, x_3 x_4, x_4^2)$, thus $\nu(0) = 3$.

*Example 3:* Let $S = \{(u, v, w) \in \mathbf{C}^3 \mid v^2 = uw\}$ and $\pi : \mathbf{C}^2 \to S$ be the map $(x, y) \to (x^2, xy, y^2)$. This map identifies $S$ with the quotient of $\mathbf{C}^2$ by the equivalence relation idenfying $(x, y)$ with $(-x, -y)$. However, $\pi$ is not flat, since for $s \in S$, $\nu(s) = 2$ if $s \neq 0$ and $\nu(s) = 3$ if $s = 0$.

B) *Projection of a product of analytic spaces*

*Theorem 2:* Let $S$ and $X$ be analytic spaces. If $\pi : S \times X \to S$ is the projection morphism, then $\pi$ is flat, i.e. $\mathcal{O}_{S\times X,(s,x)}$ is a flat $\mathcal{O}_{S,s}$ module for every $(s, x) \in S \times X$.

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when $S$ is a manifold, and finally in the general case.

(a) *Koszul complex*

Let $A$ be a ring, $M$ an $A$-module and $h_1, \ldots, h_n$ homomorphisms $M \to M$, which commute with each other, i.e. $h_i h_j = h_j h_i$ for every $i, j$.

If $1 \leqq k \leqq n$, set $Q_k = M/h_1(M) + \ldots + h_k(M)$, and $Q_0 = M$, thus, in particular, $Q_n = Q = M/\sum_{i=j}^{n} h_i(M)$,. Every $h_k$ induces a map $\tilde{h}_k\ Q_{k-1} \to Q_{k-1}$.

*Definition 2:* The sequence $(h_1, \ldots, h_n)$ is called regular if each of the mappings $\tilde{h}_k$ $(1 \leqq k \leqq n)$ is injective.

The Koszul complex of the module $M$ and of the mappings $h_k$ $(1 \leqq k \leqq n)$ $K. = K.[M; h_1, \ldots, h_n]$ is defined in the following way:

$$K_i = \wedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \qquad 0 \leqslant i \leqslant n .$$

We define the homorphisms $d_i : K_i \to K_{i-1}$ $(i > 0)$ by $\lambda \otimes x \to \sum_i (e_i \wedge \lambda) \otimes$ $\otimes h_i(x)$, where $(e_i)$ is the natural base of $A^n$. We also define $\varepsilon : K_0 \to Q$ as the natural map : $K_0 = M \to M/\sum_{i=1}^{n} h_i(M) = Q$. Using the fact that $h_1, \ldots, h_n$ commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also $\varepsilon d_1 = 0$. Thus $K.$ is really a complex.

*Theorem 3* (Poincaré-Koszul).

If $(h_1, \ldots, h_n)$ is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & if \quad i = 0 \\ 0 & if \quad i > 0 \end{cases} .$$

If $h_i \in A$, it defines the map: $A \xrightarrow{h_i I} A$, which we denote also by $h_i$. We say that $(h_1, ..., h_n)$ is a regular sequence of elements if $(h_1 I, ..., h_n I)$ is a regular sequence.

*Corollary.* If $(h_1, ..., h_n)$ is a regular sequence of elements, then the Koszul complex $K. = K. [A; h_1, ..., h_n] = \{ \wedge^{n-1} A^n \simeq A^{\binom{n}{i}} \}$ is a free resolution of $Q = A/(h_i)$ ($(h_i)$ is the ideal generated by $h_1, ..., h_n$)).

*Example:* If $A = \mathbf{C}\{x_1, ..., x_n\}$; $h_i = x_i$, then $Q_k = A/(x_1, ..., x_k) = \mathbf{C}\{x_{k+1}, ..., x_n\}$ and $Q = Q_n = \mathbf{C}$. The complex $K. = K. [A; x_1, ..., x_n]$ is a free resolution of $\mathbf{C}$.

(b) *Proof of theorem 2, when S is a complex manifold*

In this case we can take $\mathcal{O}_{S,s} = \mathbf{C}\{t_1, ..., t_m\} = A$ and if $\mathcal{O}_{X,x} = \mathbf{C}\{x_1, ..., x_n\}/(f_1, ..., f_p)$, then

$$\mathcal{O}_{S\times X,(s,x)} = \mathbf{C}\{t_1, ..., t_m, x_1, ..., x_n\}/(f_1, ..., f_p) = B .$$

$B$ is an $A$-module in a natural way.

By the corollary of the Poincaré-Koszul theorem $K. = K. [A; t_1, ..., t_m]$ in a free resolution of $\mathbf{C}$. We want to compute the modules $\mathrm{Tor}_i^A(\mathbf{C}, B) = H_i(K.\otimes B)$ $(i>0)$.

It's easily seen, that we can consider the complex $K.\otimes B$ as a Koszul complex $K'. = K. [B; t_1, ..., t_m]$ (where $t_i : B \xrightarrow{t_i I} B$). But now the sequence $(t_1, ..., t_m)$ is regular, thus by the Poincaré-Koszul theorem $H_i [K'.] = 0$ if $i>0$.

In particular: $\mathrm{Tor}_1^A(\mathbf{C}, B) = H_1 [K.\otimes B] = H_1 [K'.] = 0$. By the second flatness criterion $B$ is A-flat.

(c) *The general case*

The question being local, we can suppose that $S \subset W \subset \mathbf{C}^n$, where $W$ is open, and $S$ an analytic subspace of $W$. Let $S$ be defined by $g_1,...,g_r$. Then $S\times X \subset W\times X$ and $\mathcal{O}_S = \mathcal{O}_W/(g_1, ..., g_r)$. On the other hand $\mathcal{O}_{S\times X} = \mathcal{O}_{W\times X}/(g_1, ..., g_r) = \mathcal{O}_S \otimes_{\mathcal{O}_W} \mathcal{O}_{W\times X}$. The last equality follows from the fact, that if $\pi : X \to S$ is a morphism, and $S' \subset S$ a subspace, $X' = \pi^{-1}(S')$, then $\mathcal{O}_{X'} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X$.

*Remark :* This a particular case of the following proposition: if $\pi$ and $\pi'$ are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & S & \end{array} \qquad \mathcal{O}_{X \underset{S}{\times} Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_y \, .$$

We have proved that $\mathcal{O}_{W \times X}$ is $\mathcal{O}_W$-flat, so by scalar extension $\mathcal{O}_{S \times X}$ is $\mathcal{O}_S$ flat.

*Corollary :* If $X$ and $S$ are two manifolds and $\pi : X \to S$ is a submersion, then $\pi$ is flat.

## III. Privileged polycylinders

### § 1. *Banach vector bundles over an analytic space*

Let $E$ be a Banach space and $X$ an analytic space. We denote then by $E_X$ the trivial bundle $X \times E$ over $X$.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from $X$ to $E$. If $U \subset \mathbf{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from $U$ into $E$ consists of all functions $g : U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now $X'$ be a local model for $X$, i.e. $X'$ is the support of the quotient sheaf $\mathcal{O}_U / J$, where $U \subset \mathbf{C}^n$ is open and $J$ is a coherent sheaf of ideals of $\mathcal{O}_U$, then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathcal{H}(V, E) / J_V \, . \, \mathcal{H}(V, E)$ ($V \subset U$, $V$-open).

*Remark :* If $X'$ is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from $X'$ to $E$ which are locally induced by analytic functions on open sets in $U$.

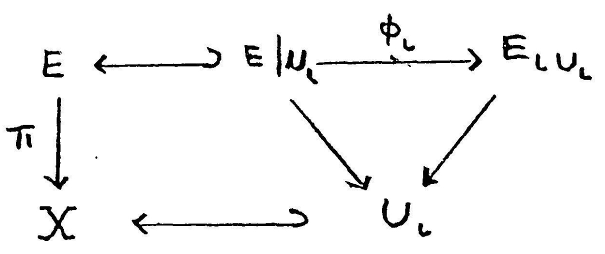
The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models $X'$ of $X$, i.e. $\mathcal{H}_X(E) | X' = \mathcal{H}_{X'}(E)$, for every local model $X'$.

*Definition 1 :* The set of *analytic morphisms* from an analytic space $X$ into a Banach space $E$ is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space $E$ into the Banach space $F$.

*Definition 2 : An analytic vector bundle morphism* from $E_X$ into $F_X$ is an analytic morphism from $X$ into $\mathcal{L}(E, F)$.

Let $E$ be a topological space, $X$ an analytic space, and $\pi : E \to X$ a continuous projection.




Suppose that $X$ has an open covering $(U_\iota)_{\iota \varepsilon I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E_{\iota U_\iota}$ and a homeomosphism $\phi_\iota$, such that the following diagram is commutative:

We suppose further that for each pair $\iota, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa} : E_{\kappa U_\iota \cap U_\kappa} \to E_{\iota U_\iota \cap U_\kappa}$, with the underlying mapping $\phi_\iota \circ \phi_\kappa^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\,\gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \qquad \text{for all} \quad \iota, \kappa, \gamma \in I\,.$$

This data gives a Banach vector bundle atlas on $E$ and provides $E$ with the structure of a Banach vector bundle over $X$ (two atlases are equivalent if there exists an atlas containing both).

*Remark:* If $X$ is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\,\gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

*Proposition 1:* Let $\phi : E \to F$ (over $X$) be a morphism of two Banach vector bundles $E$ and $F$, and $x \in X$.

If $\phi_x \in \mathscr{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of $x$, such that $\phi|U : E|U \to F|U$ is a vector bundle isomorphism.

*Proof:* First we take a trivialisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x \in V \subset X$ ($V$-open).

The set $\mathrm{Isom}(E_0, F_0)$ of isomorphic mappings is an open subset of $\mathscr{L}(E_0, F_0)$ and the mapping $g \to q^{-1}$ is an analytic isomorphism:

$$\mathrm{Isom}(E_0, F_0) \simeq \mathrm{Isom}(F_0, E_0)\,.$$

So we have in an open neighbourhood $U \subset X$ of $x$ an analytic morphism $y \to \phi_y^{-1} \in \mathscr{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|U)^{-1} : F|U \to E|U$.

*Definition 3:* Let $E$ and $F$ be two Banach spaces and $f$ a continuous linear mapping from $E$ into $F$. $f$ is a *split mono-(epi) morphism*, if there exists a mapping $g \in \mathscr{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

*Definirion 4:* Let $E_1$ and $E_2$ be two Banach vector bundles over an analytic space $X$, and $f$ a vector bundle morphism from $E_1$ into $E_2$. $f$ *is a split mono (epi) morphism*, if there exists a vector bundle morphism $g: E_2 \to E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f: E_1 \to E_2$ (over $X$) is a split monomorphism if an only if $E_2$ can be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \to G_2 \end{cases}.$$

and $f$ is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that} \quad f: \begin{cases} F_1 \to 0 \\ G_1 \simeq E_2 \end{cases}.$$

*Proposition 2:* Let $E \xrightarrow{\varphi} F$ (over $X$) be a bundle morphism and $x \in X$.

If $\phi_x: E(x) \to F(x)$ is a split epi (mono) morphism, then the point $x$ has an open neighbourhood $U \subset X$, such that $\phi|U: E|U \to F|U$ is a split vector bundle epi (mono) morphism.

*Proof:* Suppose that $\phi_x$ is a split epimorphism. We take first a trivilisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x$, so that there exists a mapping $\sigma \in \mathscr{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi: F_{0V} \to E_{0V}$ by $x \to \sigma \in \mathscr{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi: F_{0V} \to F_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in $x$. By proposition 1 we have an isomorphic restriction $\gamma|U$, $\phi|U \circ (\psi|U \circ (\gamma|U)^{-1}) = I_{F_{0U}}$.

When $\phi_x$ is a split monomorphism, the proof is similar.

*Definition 5:* Let $B_1$, $B_2$, $B_3$ be Banach spaces, and $j, k: B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms *a complex*, if $k \circ j = 0$. This sequence is *split exact* if the space $B_i$ can be decomposed in direct sums $B_i = C_i \oplus D_i$ such that

$$j : \begin{cases} C_1 \to 0 \\ D_1 \simeq C_2 \end{cases} \qquad k : \begin{cases} C_2 \to 0 \\ D_2 \simeq C_3 \end{cases} .$$

*Definition 6:* A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

is *a complex* if $g \circ f = 0$.

The sequence is *split exact*, if every $E_i$ can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_2 \end{cases} \qquad g : \begin{cases} F_2 \to 0 \\ G_2 \simeq F_3 \end{cases} .$$

*Theorem 1:* Let
$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$
be a complex of Banach vector bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \xrightarrow{fx_0} E_2(x_0) \xrightarrow{fx_0} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of $x_0$, such that $E_1|U \xrightarrow{f|U} E_2|U \xrightarrow{g|U} E_3|U$ is a split exact sequence of Banach vector bundles.

*Proof:* We take a neighbourhood $V$ of $x$, such that we have a complex $E_{1V} \xrightarrow{f|V} E_{2V} \xrightarrow{g|V} E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0} : \begin{cases} F_1(x_0) \to 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \qquad g_{x_0} : \begin{cases} F_2(x_0) \to 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2, $f|V : G_{1V} \to E_{2V}$, $g|V : G_{2V} \to E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of $x_0$ and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of $E_{2W}$ esp. $E_{3W}$, such that

$$E_{2W} = F_2 \oplus G_{2W}, \qquad E_{3W} = F_3 \oplus G_{3W} .$$

By our construction

$$g \mid W : \begin{cases} F_2 & \to 0 \\ G_2\, W \simeq F_3 \end{cases} .$$

If $p\colon E_{2W} \to F_2$ is the projection with kernel $G_{2W}$, the map, $p \circ f\colon E_{1W} \to F_2$ is a split epimorphism in $x_0$. Again by prop. 2 we have over an open eighbourhood $U \subset W$ of $x_0$ a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \operatorname{Ker} p \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 & \to 0 \\ G_{1U} \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image $f|U\,(F_1)$ is contained in $G_{2U}$. But $g|U \circ f|U = 0$ and $g|G_{2U}$ is a monomorphism hence $f|U : F_1 \to 0$. We get finally (restricting all our morphisms to $U$)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ G_{2U} \xrightarrow{\sim} F_{3U} \end{cases} .$$

## § 2. *Privileged polycylinders*

*Definition 1: A polycylinder* in $\mathbf{C}^n$ is a compact set $K$ of the form $K = K_1 \times \ldots \times K_n$ where each $K_i$ is a compact, convex subset of $\mathbf{C}$, with nonempty interior. If each $K_i$ is a disc, then $K$ is a polydisc. We first recall the following theorem of Cartan.

*Theorem 1:* Let $K$ be a polycylinder contained in an open subset $U$ of $\mathbf{C}^n$. Let $\mathscr{F}$ be a coherent analytic sheaf on $U$.

(A) There exists an open neighbourhood of $K$ over which $\mathscr{F}$ admits a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 .$$

(B) $H^q(K, \mathscr{F}) = 0$ for $q > 0$.
(Reference: For instance Gunning and Rossi.)
We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

of a coherent sheaf $\mathscr{F}$, the sequence

$$0 \to \mathscr{L}_n(K) \to \ldots \to \mathscr{L}_0(K) \to \mathscr{F}(K) \to 0$$

is an $\mathcal{O}_U(K)$ - free resolution of $\mathscr{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

then the sequence

$$0 \to \mathscr{F}_{\prime}(K) \to \mathscr{F}(K) \to \mathscr{F}''(K) \to 0 \quad \text{is exact.}$$

Let $\mathscr{F}$ be a coherent analytic sheaf on $U$, and let $K \subset U$ be a polycylinder If $V$ is an open neighbourhood of $K$, then $\mathscr{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathscr{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathscr{F}(K)$ and by choosing $K$ in a " privileged " way.

Let $B(K) = \{f : K \to \mathbf{C} | f$ continuous on $K$ and analytic on $\mathring{K}\}$, then $B(K)$ is Banach algebra and $B(K) \subset C(K)$. The sections of $\mathcal{O}_U$ over $K$ are elements of $B(K)$, and $B(K)$ is in fact the uniform closure of $\mathcal{O}_U(K)$ in $C(K)$.

If $\mathscr{L} = \mathcal{O}_U^r$, we define $B(K, \mathscr{L}) = B(K)^r$. Then $B(K; \mathscr{L})$ is a free $B(K)$-module, and since $\mathscr{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathscr{L}) = B(K) \otimes_{\mathcal{O}_{U(K)}} \mathscr{L}(K)$.

We now assume that $\mathscr{F}$ is a coherent sheaf on $U$, where $U \subset \mathbf{C}^n$ is open. Consider a free resolution

$$(R) \qquad 0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 \quad \text{of } \mathscr{F}.$$

From $(R)$ we get an $\mathcal{O}_U(K)$-free resolution of $\mathscr{F}(K)$

$$(R') \qquad 0 \to \mathscr{L}_n(K) \to \ldots \to \to_1(K) \to \mathscr{L}_0(K) \to \mathscr{F}(K) \to 0 .$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_U(K)}$ we get the complex

$$B(K; \mathscr{L}.) : 0 \to B(K; \mathscr{L}_n) \to \ldots \to B(K; \mathscr{L}_1) \to B(K; \mathscr{L}_0) .$$

*Definition 2:* The polycylinder $K$ is called *$\mathscr{F}$-privileged* if the complex $B(K; \mathscr{L}.)$ is split-exact in every degree $>0$.

*Remark:* The property of being $\mathscr{F}$-privileged is independent of the resolution $(R)$.

The exactnes of $B(K; \mathscr{L})$ can be expressed by $\mathrm{Tor}_i^{\mathcal{O}(K)}(B(K), \mathscr{F}(K)) = 0$, for every $i > 0$, and Tor is independent of the resolution $(R)$. It is a little more complicated to show, that the splitting property is independent of $(R)$, and this is omitted.

Since $B(K; \mathcal{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if $K$ is $\mathcal{F}$-privileged. In this case we define $B(K; \mathcal{F}) = \operatorname{Coker}(B(K, \mathcal{L}_1) \to B(K; \mathcal{L}_0)) = B(K) \underset{\mathcal{O}_U}{\otimes} \mathcal{F}(K)$ and we get a $B(K)$-module, which is a Banach-space.

*Warning:* In the definition of split-exactnes, the subspaces are splitting vector spaces, but they are not splitting $B(K)$-modules in general.

We have the following important theorem about the existence of privileged polycylinders:

*Theorem 2:* Let $U$ be an open subset of $\mathbf{C}^n$, and let $\mathcal{F}$ be a coherent analytic sheaf on $U$. For any $x \in U$ there exists a fundamental system of neighbourhoods of $x$ in $U$, which are $\mathcal{F}$-privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

*Example:* (Curves in $\mathbf{C}^2$) Let $U \subset \mathbf{C}^2$ be an open connected neighbour hood of the origin, and let $h\colon U \to \mathbf{C}$ be analytic and $h \neq 0$.

Let $X$ be the curve given by $h$, that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U/(h)$. We have an exact sequence $0 \to \mathcal{O}_U \xrightarrow{h.I} \mathcal{O}_U \to \mathcal{O}_X \to 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition $K$ is $\mathcal{O}_X$-priviledged if and only if $h\colon B(K) \to B(K)$ is a split monomorphism.

Let $\dot{K}_j$ denote the boundary of $K_j$, and define $\ddot{K} = \dot{K}_1 \times \dot{K}_2$ ($\ddot{K}$ is called the Šilov Boundary of $K$).

*Proposition 1:* (*a*) The following conditions are equivalent:

(i) $h\colon B(K) \to B(K)$ is a monomorphism.

(i′) $\exists\, a > 0$ such that $\|hf\| \geqq a\|f\|$, $\forall f \in B(K)$.

(ii) $X \cap \ddot{K} = \varnothing$.

(*b*) If $(K_1 \times \dot{K}_2) \cap X = \varnothing$, then $h$ is a split monomorphism (i.e. $K$ is $\mathcal{O}_X$ privileged).

*Proof:* (a) (i) $\Leftrightarrow$ (i′) is a well known fact from the theory of normed vector spaces.

(ii) $\Rightarrow$ (i′). Assume $X \cap \ddot{K} = \varnothing$. If $f \in B(K)$, then it follows from the maximum principle that $\|f\| = \sup\limits_{K} |f(x)| = \sup\limits_{\ddot{K}} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in \ddot{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $||hf|| = \sup_K |hf(x)| \geqq$ $\geqq a \sup_K |f(x)| = a\, ||f||$.

(i′) ⇒ (ii). Suppose that $X \cap \ddot{K} \neq \varnothing$ and $x = (x_1, x_2) \in X \cap \ddot{K}$. We choose an analytic function $f_1 : U_1 \to \mathbf{C}$, where $U_1 \supset K_1$, and $U_1$ is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in $K$.

Applying Dini's theorem we get $||hf^n|| \to 0$. From the inequality $a\,||f^n|| \leqq$ $\leqq ||hf^n||$ we get $||f^n|| \to 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

*Question.* Does the condition (ii) imply that $h : B(K) \to B(K)$ is a split monomorphism?

## IV. Flatness and privilege

### § 1. *Morphisms from an analytic space into $B(K)$*

Let $S$ be an analytic space and $K$ a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an $\mathcal{O}_S$-algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \to$ $\to \mathcal{H}(S; B(K))$.

(*a*) Consider first $S = U' \subset \mathbf{C}^m$, $U'$-open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that $\phi$ is an $\mathcal{O}_{U'}$-algebra homomorphism.

(*b*) Let $S$ have a special model in the polydisc $\Delta$ in $\mathbf{C}^m$, defined by a sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_\Delta$, and let $\mathcal{J}$ be generated by $f_1, \ldots, f_p$, $V$-a polycylinder neighbourhood of $K$ in $U$. By Cartan's theorem $B$ for a polycylinder,

the sequence $0 \to \mathcal{J}(\Delta \times V) \to \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \to 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \to \mathcal{H}(S, B(K))$, $(f_1, \ldots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$ $\subset \operatorname{Ker} \tilde{\pi}$. Therefore, because $\pi$ is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \to \mathcal{H}(S, B(K))$, such that the diagram

$$\begin{array}{ccc} \mathcal{O}(\Delta \times V) & \overset{\phi}{\rightarrow} & \mathcal{H}(\Delta, B(K)) \\ \pi\downarrow & & \downarrow\tilde{\pi} \\ \mathcal{O}(S \times V) & \overset{\phi}{\rightarrow} & \mathcal{H}(S, B(K)) \end{array}$$

is commutative; $\phi$ is evidently an $\mathcal{O}_S$-algebra homomorphism.

## § 2. *The flatness and privilege theorem*

*Notation*

Let $S$ be an analytic space, $U$ an open set in $\mathbf{C}^n$, and $\pi : S\times U \to S$ the first projection.

If $\mathcal{F}$ is an $\mathcal{O}_{S\times U}$ module, then for every $s \in S$ we denote by $\mathcal{F}(s)$ the $\mathcal{O}_U$-module $i_s^* \mathcal{F}$, where $i_s$ is the injective morphism $x \to (s, x)$ from $U$ into $S\times U$. If $x \in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s,x)}/m_s \cdot \mathcal{F}_{(s,x)} \simeq \mathcal{F}_{(s,x)} \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s \,.$$

*Theorem 1:* Let $\mathcal{E}$ be a coherent and $S$-flat $\mathcal{O}_{S\times U}$-module, and $K$ a polycylinder in $U$.

(*a*) When $K$ is privileged for $\mathcal{E}(s_0)$, $s_0$ has a neighbourhood $V$ such that $K$ is $\mathcal{E}(s)$-privileged for each $s \in V$. In other words: the set $S' = \{s \in S \mid K$ is $\mathcal{E}(s)$-privileged$\}$ is open in $S$.

(*b*) It is possible to define a Banach vector bundle over $S'$ whose fibre at any $s \in S'$ is $B(K, \mathcal{E}(s))$.

To prove the theorem we need:

*Lemma 1:* Under the conditions of the theorem, we can, for every $s \in S$, find a neighbourhood $W$ of $\{s\}\times K$ and a free resolution of finite length

$$0 \to \mathcal{L}_p \overset{d_p}{\to} \dots \overset{d_2}{\to} \mathcal{L}_1 \overset{d_1}{\to} \mathcal{L}_0 \overset{\varepsilon}{\to} \mathcal{E} \to 0 \text{ in } W.$$

*Proof:* Let $(s, x)$ be a point of $S\times U$ and $\mathcal{L}^0_*$ a finite resolution of $\mathcal{F}(x)$ in a neighbourhood of $x$ (there exists such one, by the theorem of syzygies). We shall show that that there exists a resolutin $\mathcal{L}^*$ of $\mathcal{F}$ in a neighbourhood of $(s, x)$ such that $\mathcal{L}^*(s) = \mathcal{L}^0_*$; if $\mathcal{L}^0_i = \mathcal{O}_x^{r_i}$ define

$$\mathcal{L}_i = \mathcal{O}^{r_i}_{S\times U} \text{ and } \mathcal{K}^0_i = \operatorname{Ker} d^0_i\colon \mathcal{L}^0_i \to \mathcal{L}^0_{i-1}.$$

We shall construct by induction (with respect to $i$) $d_i : \mathscr{L}_1 \to \mathscr{L}_{i-1}$ in a neighbourhood of $(s, x)$ such that $d_i(s) = d_i^0$, and prove that $\mathscr{K}_i = \operatorname{Ker} d_i$ is $S$-flat and that $\mathscr{K}_i(s) = \mathscr{K}_i^0$.

$$\begin{array}{ccc} \mathscr{L}_{i+1} & \xrightarrow{d_{i+1}} & \mathscr{K}_i \\ \downarrow & & \downarrow \\ \mathscr{L}_{i+1}^0 & \xrightarrow{d_{i+1}} & \mathscr{K}_i^0 \end{array}$$

Suppose that we have constructed $d_i$ and proved the properties for $\mathscr{K}_i$. We can construct $d_{i+1} : \mathscr{L}_{i+1} \to \mathscr{L}_i$ in a neighbourhood of $(s, x)$ such that the diagram is commutative.

Nakayama's lemma shows that $\operatorname{Im} d_{i+1} = \mathscr{K}_i$ at the point $(s, x)$, therefore in a neighbourhood of that point.

The exact sequence

$$0 \to \mathscr{K}_{i+1} \to \mathscr{L}_{i+1} \to \mathscr{K}_i \to 0\,,$$

where $\mathscr{K}_i$ and $\mathscr{L}_{i+1}$ are $S$-flat, shows that $\mathscr{K}_{i+1}$ is $S$-flat, and that $\mathscr{K}_{i+1}(s) = \mathscr{K}_{i+1}^0$. The first step of the induction is analogous.

*Proof of the theorem:* Let $s_0 \in S$ and

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \ldots \xrightarrow{d_1} \mathscr{L}_0 \to \mathscr{E} | W \to 0$$

be a free $\mathscr{O}_{S\times U}$ resolution of $\mathscr{E}$ in a neighbourhood $W = V_1 \times V_2$ of $\{s_0\} \times K$. The sheaf $\mathscr{E}$ is $\mathscr{O}_S$-flat, so for each $s \in V_1$, the sequence

$$0 \to \mathscr{L}_p \otimes_{\mathscr{O}_{S|V_1}} \mathrm{C}_s \to \ldots \to \mathscr{L}_1 \otimes_{\mathscr{O}_{S|V_1}} \mathrm{C}_s \to \mathscr{L}_0 \otimes_{\mathscr{O}_{S|V_1}} \mathrm{C}_s \to \mathscr{E}_{|W} \otimes_{\mathscr{O}_{S|V_1}} \mathrm{C}_s \to 0$$

is exact. So the sequence

$$(A) \qquad 0 \to \mathscr{L}_p(s) \xrightarrow{d_p(s)} \ldots \to \mathscr{L}_1(s) \xrightarrow{d_1(s)} \mathscr{L}_0(s) \xrightarrow{\varepsilon(s)} \mathscr{E}(s)_{|V_2} \to 0$$

is exact when $s \in V_1$. Now $\mathscr{L}_i(s) \simeq \mathscr{O}_{V_2}^{r_i}$ $(0 \leq i \leq p)$ and every $d_i(s)$ induces a continuous linear map:

$B(K, \mathscr{L}_i(s)) \to B(K, \mathscr{L}_{i-1}(s))$, which we also denote by $d_i(s)$. We can consider $d_i = (d_{ijk})$ as an $r_i \times r_{i-1}$-matrix with entries from $\mathscr{O}_{S\times U}(W)$.

By § 1 we have a $\mathscr{O}_S$-algebra homomorphism

$$\mathscr{O}_{S\times W}(S \times W) \to \mathscr{H}(S, B(K))\,.$$

From the matrix $(d_{ijk})$ we get by this homomorphism a morphism $\tilde{d}_i$:

$$V_0 \to \mathscr{L}(B(K)^{ri}, B(K)^{ri-1}) = \mathscr{L}(B(K, \mathscr{L}_i(s)), B(K, \mathscr{L}_{i-1}(s))\,.$$

(Here $V_0$ is some neighbourhood of $s_0$) such that $\tilde{d}_i(s) = d_i(s)$ for each $s \in V_0$. In other words we have a sequence of Banach vector bundle morphisms

$$(B) \qquad 0 \to B(K, \mathscr{L}_p) \xrightarrow{d_p} \dots \xrightarrow{\tilde{d}_1} B(K, \mathscr{L}_0).$$

Using the fact that $\mathscr{O}_{S\times U}(S\times U) \to \mathscr{H}(S, B(K))$ is an $\mathscr{O}_S$-algebra homomorphism, it easily follows that $(B)$ is complex of Banach vector bundles over $S$.

Now $K$ is $\mathscr{E}(s_0)$-privileged, thus

$$0 \to B(K, \mathscr{L}_p(s_0)) \xrightarrow{d_p(s_0)} \dots \xrightarrow{d_1(s_0)} B(K, \mathscr{L}_0(s_0))$$

is split exact, so by theorem III.1

$$0 \to B(K, \mathscr{L}_p)_{|V} \xrightarrow{\tilde{d}_p|V} \dots \xrightarrow{\tilde{d}_i|V} B(K, \mathscr{L}_0)_{|V}$$

is split exact for some neighbourhood $V$ of $s_0$.

Because $\tilde{d}_i(s) = d_i(s)$ and the sequence $(A)$ is exact part $(a)$ of the theorem follows.

$(b)$ $B(K. \mathscr{L}_0)|V$ splits as the direct sum of im $\tilde{d}_1$ and a bundle $E_V$, such that $E_{V,s} \simeq B(K, \mathscr{E}(s))$, for each $s \in V$. We must show that these bundle structures fit together globally.

Suppose therefore that $V$ is open in $S'$ and that

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathscr{L}_1 \xrightarrow{d_1} \mathscr{L}_0 \to \mathscr{E}_{|V\times V_2} \to 0$$

$$0 \to \mathscr{L}'_p \xrightarrow{d'_p} \dots \xrightarrow{d'_2} \mathscr{L}'_1 \xrightarrow{d'_1} \mathscr{L}'_0 \to \mathscr{E}_{|V\times V_2} \to 0$$

are free resolutions of $\xi$ over $V \times V_2$.

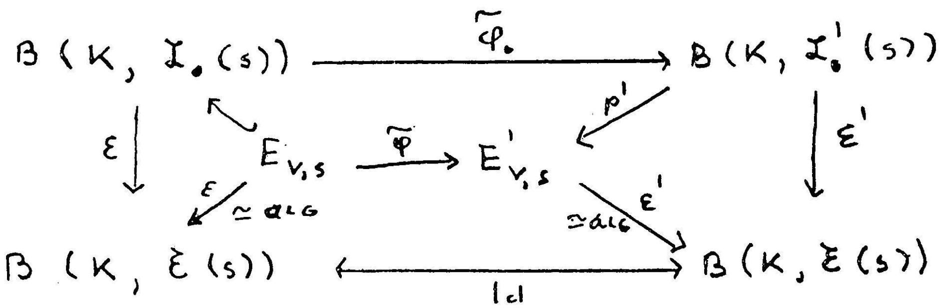
If $V_1$, $V_2$ are open polycylinders, we can find an $\mathscr{O}_{S\times U}$-homomorphism $\phi_0 : \mathscr{L}_0 \to \mathscr{L}'_0$ such that

$$\begin{array}{ccccc} \mathscr{L}'_0 & \xrightarrow{\varepsilon'} & \mathscr{E}_{|V\times V_2} & \to & 0 \\ \phi_0 \uparrow & & \| & & \\ \mathscr{L}_0 & \xrightarrow{\varepsilon} & \mathscr{E}_{|V\times V_2} & \to & 0 \end{array}$$

commutes. $\phi_0$ determines a bundle morphism $\tilde{\phi}_0$: $B(K, \mathscr{L}_0) \to B(K, \mathscr{L}'_0)$. $B(K, \mathscr{L}_0)$ (resp. $B(K, \mathscr{L}'_0)$) splits as (im $\tilde{d}_1$)$\otimes E_V$ [Resp. (im $\tilde{d}'_1$)$\otimes E'_V$].

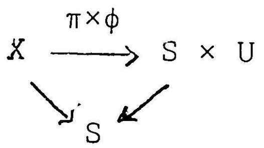
Let $p'$ be the projection morphism: $B(K, \mathscr{L}_0) \to E'_V$ with kernel im $\tilde{d}'_1$, and put $\tilde{\phi} = p' \circ \tilde{\phi}_0 | E_V$.

The commutative diagram




and the open mapping theorem shows that $\tilde{\phi}(s)$ is an isomorphism of Banach spaces for each $s \in V$, so $\tilde{\phi}: E_V \to E'_V$ is a bundle isomorphism. We also notice that $\tilde{\phi}$ depends only on the choice of splittings in $B(K, \mathscr{L}_0)$ and $B(K, \mathscr{L}'_0)$, and not on the choice of $\tilde{\phi}_0$. This ends the proof of the theorem.

*Remark:* Consider the general situation where $X$ and $S$ are analytic spaces, and $\pi: X \to S$ is a morphism, $\mathscr{E}$ an $\mathscr{O}_X$-module. To study the local dependence of $\mathscr{E}$ on $S$, one can imbed an open set $X'$ in $X$ in the open set $U \subset \mathbf{C}^n$. The morphism $\phi: X' \to U$, $\pi: X' \to S$ determine the imbedding $\pi \times \phi: X' \to S \times U$ such that the diagram




commutes. $\mathscr{E}$ can be extended by zero into a sheaf $\mathscr{E}'$ over $U \times S$. Obviously this sheaf $\mathscr{E}'$ is $S$-flat iff $\mathscr{E}$ is $S$-flat.

Therefore theorem 1 makes clear also this general situation.

*Corollary:* If $\pi: X \to S$ is a morphism and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Then $\pi \mid \mathrm{Supp}(\mathscr{E})$ is an open map.

*Proof:* Suppose as above that $X$ is imbedded in $S \times U$, and $\mathscr{E}$ in extended by zero to $S \times U$. Let $x_0 \in \mathrm{Supp}\, \mathscr{E}$, and $V$ be a neighbourhood of $x_0$ in $S \times U$. Let $s_0 = \pi(x_0)$ and choose an $\mathscr{E}(s_0)$-privileged polycylinder $K$ in $U$, such that $\{s_0\} \times K \subset V$, over some neighbourhood $W$ of $s_0$. We have the Banach bundle $B(K, \mathscr{E} | \pi^{-1}(W))$, whose fiber over $s$ is $B(K, \mathscr{E}(s))$. Since $x_0 \in \mathrm{Supp}\, \mathscr{E}(s_0)$ and $K$ is a neighbourhood of $x_0$, $B(K; \mathscr{E}(s_0)) \neq 0$. As all the fibers are isomorphic, then for all $s \in U$, $B(K; \mathscr{E}(s)) \neq 0$ and therefore $\{s\} \times K \cap \mathrm{Supp}\, \mathscr{E} \neq 0$, and $s \in \pi(\mathrm{Supp}\, \mathscr{E})$. This proves that $\pi$ is open.

## REFERENCES

GUNNING, R. C. and H. ROSSI, *Analytic functions of several complex variables.* Prentice-Hall. Inc. Englewood Cliffs, N.J. 1965.

DOUADY, A., Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. *Annales de Fourier*, *Université de Grenoble*, tome XVI, 1966, pp. 1-95.

MALGRANGE, B., *Lecture Notes.* Analytic spaces (see this volume, p. 1-28)