

# FLATNESS AND PRIVILEGE

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# FLATNESS AND PRIVILEGE

by A. DOUADY

## I. FLAT MORPHISMS

### § 1. Analytic subspaces of an analytic space

Let  $Y_1$  and  $Y_2$  be closed analytic subspaces of an analytic space  $X$ , and let them be defined by the  $\mathcal{O}_X$  ideals  $J_1, J_2$ .

*Definition 1:* We say that  $Y_1$  is *analytically included* in  $Y_2$ , and we write  $Y_1 \subset Y_2$ , when  $J_1 \supset J_2$ .

*Remark:* The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example:  $X = (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$ ;  $J_1 = (x)$ ,  $J_2 = (x^2)$ . The space  $Y_1$  is a simple point,  $Y_2$  is a double point,  $Y_1 \not\subset Y_2$ , while they have the same underlying set.

*Definition 2:* The subspace  $Y_1 \cup Y_2$  is the smallest subspace of  $X$  containing  $Y_1$  and  $Y_2$ , and it is defined by  $J_1 \cap J_2$ . The subspace  $Y_1 \cap Y_2$  is the biggest subspace of  $X$  contained in both  $Y_1$  and  $Y_2$ , and it is defined by  $J_1 + J_2$ .

*Remark:* The underlying set of  $Y_1 \cup Y_2$  (Resp.  $Y_1 \cap Y_2$ ) is the union (Resp. intersection) of the underlying sets of  $Y_1$  and  $Y_2$ . However  $\cup$  and  $\cap$  of analytic spaces do not satisfy the distributivity laws which hold in set-theory:  $(Y_1 \cup Y_2) \cap Y_3$  contains  $Y_1 \cap Y_3$  and  $Y_2 \cap Y_3$ , and therefore their union; similarly  $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$ . In general the converse inclusions do not hold.

Example: Let  $X = \mathbf{C}^2$  and  $Y_1, Y_2, Z$  be given by ideals  $(x-y)$ ,  $(x+y)$  and  $(x)$  respectively.

$(Y_1 \cup Y_2) \cap Z$  is  $\{0\}$  provided with  $\mathbf{C}\{y\}/(y^2)$ , while  $(Y_1 \cap Y_2) \cup (Y_2 \cap Z)$  is the reduced space  $\{0\}$ . On the other hand:  $Y_1 \cap Y_2 \subset Z$ ,  $(Y_1 \cap Y_2) \cup Z = Z$ , while  $(Y_1 \cup Z) \cap (Y_2 \cup Z)$  is the space defined by the ideal  $(x^2, xy)$ . Its local ring at the origin is  $\mathbf{C}\{x, y\}/(x^2, xy)$  in which  $x$  is nilpotent.

*Definition 3 :* Let  $X', X$  be analytic spaces,  $Y$  a closed analytic subspace of  $X$  defined by  $J$ , and  $f = (f_0, f^1) : X' \rightarrow X$  a morphism.

The inverse image of  $Y$  by  $f, f^{-1}(Y)$ , is the analytic subspace  $Y'$  of  $X'$  defined by the ideal  $J' = f^1(J) \mathcal{O}_{X'}$ .

The inverse image of a simple point  $x$  in  $X$  is called the  $f$ -fiber over  $x$ , and is denoted by  $f^{-1}(x)$  or  $X'(x)$ .

*Proposition 1 :* If  $f = (f_0, f^1) : X' \rightarrow X$  is a morphism of analytic spaces, and  $Y$  is a subspace of  $X$ , then  $f^{-1}(Y) \simeq \underset{X}{Y \times X'}$ .

*Proof :* Let  $T$  be any analytic space, and  $g : T \rightarrow X'$  a morphism. Then  $g$  can be considered as a morphism from  $T$  to  $f^{-1}(Y)$  if and only if  $f \circ g$  can be considered as a morphism from  $T$  to  $Y$ . Thus  $f^{-1}(Y)$  and  $\underset{X}{X' \times X}$  are solutions of the same universal problem.

## § 2. Analytic pull-back

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product  $E \otimes_A F$ , where  $A$  is a commutative ring and  $E, F$  are two  $A$ -modules.

(1°)  $E \otimes A^n = E^n$  ( $n \in \mathbb{N}$ )

(2°) If the sequence of  $A$ -modules  $F' \rightarrow F \rightarrow F'' \rightarrow 0$  is exact, then also the sequence  $E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$  is exact. (Right exactness of the tensor product)

(3°) If  $(F_i)_{i \in I}; f_{ij} : F_j \rightarrow F_i$  is an inductive system, then

$$E \otimes \lim_{\rightarrow} F_i = \lim_{\rightarrow} (E \otimes F_i).$$

On the other hand these properties characterize completely the functor  $\otimes$ .

*Definition 1 :* Let  $f = (f_0, f^1) : X' \rightarrow X$  be a morphism of analytic spaces, and  $\mathcal{E}$  an  $\mathcal{O}_X$ -module. Then  $f_0^* \mathcal{E}$  is an  $f_0^* \mathcal{O}_X$ -module and  $\mathcal{O}_{X'}$  is also an  $f_0^* \mathcal{O}_X$ -module (by  $f^1 : f_0^* \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$ ).

The analytic pull-back  $f^* \mathcal{E}$  of  $\mathcal{E}$  by  $f$  is defined by scalar extension:

$$f^* \mathcal{E} = f_0^* \mathcal{E} \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$

*Remark* : The inverse image is a particular case of the analytic pull-back.

In fact, if  $Y$  is a closed analytic subspace of  $X$  and  $f : X' \rightarrow X$  is a morphism:

$$f^* \mathcal{O}_Y = f_0^* (\mathcal{O}_X / J_Y) \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'} / f^1(J_Y). \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that  $A/I \otimes_A E \simeq E/IE$ ).

*Elementary properties of the analytic pull-back :*

- (a)  $(f^* \mathcal{E})_{x'} = (f_0^* \mathcal{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$  where  $x = f_0(x')$  (since  $\otimes$  commutes with inductive limits).
- (b)  $f^* (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f^* \mathcal{E} \otimes_{\mathcal{O}_{X'}} f^* \mathcal{F}$ , where  $\mathcal{E}$  and  $\mathcal{F}$  are  $\mathcal{O}_X$ -modules.
- (c) If  $\mathcal{E}$  is a coherent  $\mathcal{O}_X$ -module, then  $f^* \mathcal{E}$  is a coherent  $\mathcal{O}_{X'}$ -module.

In fact,  $\mathcal{E}$  has a locally finite presentation:

$$\mathcal{O}_X^q \rightarrow \mathcal{O}_X^p \rightarrow \mathcal{E} \rightarrow 0, \text{ and } f^* \text{ is compatible with cokernels, } f^* (\mathcal{O}_X^r) = \mathcal{O}_{X'}^r.$$

*Special case* : The pull-back of vector bundle. Let  $(E, \pi)$  be an analytic

$$\begin{array}{ccc} E \times X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

vector bundle over the analytic space  $X$ , and  $f : X' \rightarrow X$  a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over  $X'$ , such that  $\bar{f}$  is a bundle morphism. We call this bundle  $E'$ .

*Proposition 1* : Let  $\mathcal{E}$  (Resp.  $\mathcal{E}'$ ) be the sheaf of analytic sections of  $E$  (Resp.  $E'$ ). Then  $\mathcal{E}' = f^* \mathcal{E}$ .

*Proof (Sketch)* : We have a  $f_0^* \mathcal{O}_X$  linear morphism  $f_0^* \mathcal{E} \rightarrow \mathcal{E}'$ , which extends to a morphism  $f^* \mathcal{E} \rightarrow \mathcal{E}'$ . We can prove that this is an isomorphism. Since the question is local with respect to  $X'$ , we can suppose that  $E$  is a trivial bundle over  $X$  with fiber  $\mathbf{C}^r$ , then  $\mathcal{E} = \mathcal{O}_X^r$ . Also  $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$ . Therefore  $f^* \mathcal{E} = \mathcal{E}'$ .

### § 3. Introduction to flatness by examples

Let  $S$  be an analytic space. By analytic space over  $s$  we mean an analytic space  $X$  provided with a morphism  $\pi : X \rightarrow S$ . Let  $S$  be a simple point in  $S$ , and consider  $X(s) = f^{-1}(s)$ .

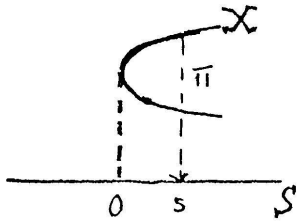


The main purpose of these lectures is to give a precise meaning to the expression:

“  $X(s)$  depends nicely on  $s$ ”, and to give a criterion for the “ nice ” behaviour.

We begin with some examples.

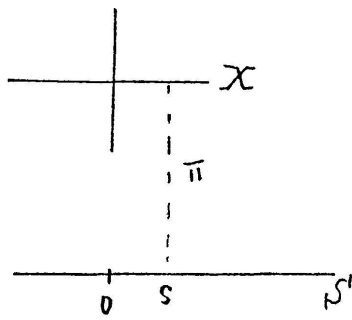
*Example 1:*  $X$  is the closed subspace on  $\mathbf{C}^2$  defined by  $(y^2 - x)$ ,  $S = \mathbf{C}$  and  $\pi = 1st$  projection.



$$X(s) = \begin{cases} 2 \text{ simple points if } s \neq 0 \\ \text{double point if } s = 0. \end{cases}$$

Here the behaviour of  $X(s)$  is nice.

*Example 2:*  $X$  is the closed subspace of  $\mathbf{C}^2$  defined by  $(xy)$ ,  $S = \mathbf{C}$  and  $\pi = 1st$  projection.



$X(s)$  is given by  $(x-s, xy)$ , and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if } s \neq 0 \\ (x) & \text{if } s = 0. \end{cases}$$

The first case is a simple point, the second one the,  $y$ -axis.

A similar example is the map of a point into  $\mathbf{C}$ .

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreducible component of  $X$ , and after removing this component  $\pi$  behaves nicely.

This kind of removing is not possible in general, as the following example shows:

*Example 3:*  $X$  is given in  $\mathbf{C}^3$  by  $(xz - y)$ , and  $\pi$  is the projection on the  $(x, y)$ -plane.

If  $s = (x_0, y_0)$ , then the fiber  $X(s)$  is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\frac{y_0}{x_0}\right) & \text{if } x_0 \neq 0 \\ (x, y) & \text{if } x_0 = y_0 = 0 \\ (1) & \text{if } x_0 = 0, y_0 \neq 0. \end{cases}$$

The set of “ nice ” fibers is dense in  $X$ , so we cannot remove the  $z$ -axis and still get a closed subspace of  $\mathbf{C}_3$ .

#### § 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

*Definition 1:* An  $A$ -module  $E$  is *flat*, if for every exact sequence of  $A$ -modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0,$$

the sequence  $0 \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$  is also exact. We can also say, because  $\otimes$  is right exact, that  $E$  is flat, if for every injective homomorphism  $F' \rightarrow F$ ,  $E \otimes F' \rightarrow E \otimes F$  is also injective.

*Examples of modules which are not flat :*

- (1) if  $A = \mathbf{Z}$ ,  $E = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ ,  $F = F' = \mathbf{Z}$ ; then the sequence  $0 \rightarrow \mathbf{Z} \xrightarrow{2I} \mathbf{Z} (2I : x' \rightarrow 2x)$  is exact. But now  $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$ , and the homomorphism  $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$  is the zero homomorphism, which is not injective. So  $\mathbf{Z}_2$  is not a flat  $\mathbf{Z}$  module.
- (2) If  $A = \mathbf{C}\{x\}$ ,  $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$ ,  $F = F' = \mathbf{C}\{x\}$ , then the sequence  $0 \rightarrow F \xrightarrow{xI} F' (xI : p(x) \rightarrow xp(x))$  is exact. But the homomorphism  $E \xrightarrow{xI} E$  is not injective.

*Proposition 1 :* If  $A$  is an integral domain and  $E$  a flat  $A$ -module, then  $E$  is torsion-free.

*Proof :* Let  $a \in A$ ,  $a \neq 0$ . Because  $A$  is an integral domain, the sequence  $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$  is exact. Since  $E$  is flat, the sequence  $0 \rightarrow E \xrightarrow{aI} E$  is also exact. In other words  $E$  has no torsion elements.

*Proposition 2 :* If  $A$  is a principal-ideal domain, then  $E$  is flat if and only if  $E$  is torsionfree.

*Proof :* See corollary of prop. 6.

*Examples of flat modules :*

- (1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if  $E$  is free and finite type, then  $E = A^n$  and  $E \otimes F = F^n$ . If  $F' \rightarrow F$  is injective, so is  $F'^n \rightarrow F^n$  too.

If  $E$  is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of  $E$  follows from (1).

(3) Let  $S$  be a multiplicative system in  $A$ . Then the ring of fractions  $S^{-1}A$  is a flat  $A$ -module. In fact the ring  $S^{-1}A$  can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that  $S$  has only regular elements. We can define in the set  $S$  a partial order in the following way:

$$s' \geq s \Leftrightarrow \exists t \in A, \quad ts = s' \quad (\text{such a } t \text{ is then unique}).$$

Let  $E_s = A$  for every  $s \in S$ , and if  $s' \geq s$  (i.e.  $s' = ts$ ) then let  $f_s^{s'}$  be the homomorphism  $t \cdot I_A : E_s \rightarrow E_{s'}$ . The family  $(E_s)_{s \in S}$  with the homomorphisms  $(f_s^{s'})$  is an inductive system.

Let  $E = \lim_{\rightarrow} E_s$  be the inductive limit of this system, and  $\varphi_s$  the canonical homomorphism  $E_s \rightarrow E$ . We shall define an isomorphism  $\psi : E \rightarrow S^{-1}A$ .

We first define for every  $s$  a homomorphism  $\psi_s : E_s = A \rightarrow S^{-1}A$ ;  $x \rightarrow x/s$ . Now if  $s' \geq s$ , then

$$(\psi_{s'} \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x).$$

Therefore there exists a homomorphism  $\psi : E \rightarrow S^{-1}A$ , satisfying  $\psi_s = \psi \circ \varphi_s$  for every  $s \in S$ .

Because every element of  $S^{-1}A$  has the form  $a/s$ ,  $\psi$  is surjective. On the other hand if  $\psi(\varphi_s(x)) = 0$ , then  $\psi_s(x) = x/s = 0$ . Thus  $x = 0$ , and  $\psi$  is also injective.

The above proof can be extended to the general case, not assuming that the elements of  $S$  are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

*Theorem 1 : (Daniel, Lazard)*

Any flat module is a inductive limit of free modules.

For the proof: See *C.R. Acad. Sci. Paris*, 258 (1964), pp. 6313-6316.

*Some elementary properties of flat modules :*

- (1) If  $E$  and  $F$  are flat  $A$ -modules, then  $E \otimes_A F$  is also flat. In fact, if  $G' \rightarrow G$  is injective, then  $F \otimes_A G' \rightarrow F \otimes_A G$  is injective, and also  $E \otimes_A (F \otimes_A G') \rightarrow E \otimes_A (F \otimes_A G)$  is injective. The result follows from the associativity of the tensor product.
- (2) Let  $\phi : A \rightarrow B$  be a ring homomorphism, and  $E$  a flat  $A$ -module. The module  $B \otimes_A E$  is a flat  $B$ -module.

If  $F$  is a  $B$ -module, then  $F \otimes_B (B \otimes_A E) = (F \otimes_B B) \otimes_A E = F \otimes_A E$  further if  $F'$  and  $F$  are  $B$ -modules, and  $F' \rightarrow F$  an injective homomorphism of  $B$ -modules, we can consider this homomorphism as an injective homomorphism of  $A$ -modules. Because  $E$  is  $A$ -flat,

$$F' \otimes_A E \rightarrow F \otimes_A E \text{ is injective.}$$

- (3) Let  $\phi : A \rightarrow B$  be a ring homomorphism, such that  $B$  is a flat  $A$ -module. If  $F$  is a flat  $B$ -module, then  $F$  is a flat  $A$ -module. In fact: if  $E' \rightarrow E$  is injective, then  $E' \otimes_A B \rightarrow E \otimes_A B$  is injective, and also  $(E \otimes_A B) \otimes_B F' \rightarrow (E \otimes_A B) \otimes_B F$  is injective. But  $(E' \otimes_A B) \otimes_B F' = E' \otimes_A F$ ;  $(E \otimes_A B) \otimes_B F = E \otimes_A F$ .

If an  $A$ -module  $E$  is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor  $\text{Tor}$ .

*Definition 2 :* A free resolution of  $E$  is an exact sequence:  $\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow E \rightarrow 0$ , where all  $L_i$  are free  $A$ -modules.

The complex of the resolution is the sequence

$$(L.) \dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow 0.$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensor products  $L_i \otimes F$ , we get

$$(L. \otimes F) \dots \rightarrow L_n \otimes F \rightarrow L_{n-1} \otimes F \rightarrow \dots \rightarrow L_1 \otimes F \rightarrow L_0 \otimes F \rightarrow 0.$$

*Definition 3 :*

$$\text{Tor}_n^A(E, F) = H_n(L. \otimes F) = \frac{\text{Ker}(L_n \otimes F \rightarrow L_{n-1} \otimes F)}{\text{Im}(L_{n+1} \otimes F \rightarrow L_n \otimes F)}$$

if  $n \geq 1$ , and  $\text{Tor}_0^A(E, F) = \text{Coker}(L_1 \otimes F \rightarrow L_0 \otimes F) = E \otimes F$ .

*Basic properties of Tor :*

- (1)  $\text{Tor}_n(E, F)$  is independent of the choice of the resolution (up to a canonical isomorphism).

- (2) If we take a free resolution of  $F$ , we get  $\text{Tor}_n(F, E) = \text{Tor}_n(E, F)$  (Symmetry of the Tor). We can also define  $\text{Tor}_n(E, F)$  by taking two free resolutions, one of  $E$  and one of  $F$ .
- (3) If  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$  is a short exact sequence, then we get a long exact sequence:

$$\begin{array}{ccccccc} \text{Tor}_n(E', F) & \rightarrow & \text{Tor}_n(E, F) & \rightarrow & \text{Tor}_n(E'', F) & \rightarrow & \\ \rightarrow & \text{Tor}_{n-1}(E', F) & \rightarrow & \text{Tor}_{n-1}(E, F) & \rightarrow & \text{Tor}_{n-1}(E'', F) & \rightarrow \\ \rightarrow & \text{---} & \rightarrow & \text{---} & \rightarrow & \text{---} & \rightarrow \\ \rightarrow & \text{Tor}_1(E', F) & \rightarrow & \text{Tor}_1(E, F) & \rightarrow & \text{Tor}_1(E'', F) & \rightarrow \\ \rightarrow & E' \otimes F & \rightarrow & E \otimes F & \rightarrow & E'' \otimes F & \rightarrow 0. \end{array}$$

- (4) Tor is compatible with inductive limit, i.e. if  $E = \lim (E_i)$ , then
- $$\text{Tor}_n(\lim E_i, F) = \lim (\text{Tor}_n(E_i, F)).$$

- (5) We can define  $\text{Tor}_n(E, F)$  by taking a flat resolution of  $E$ .

*Proposition 3:* Let  $E$  be an  $A$ -module. Then the following conditions are equivalent:

- (a)  $E$  is flat.  
 (b) For all  $A$ -modules  $F$ , and for all  $n \geq 1$ ,  $\text{Tor}_n(E, F) = 0$ .  
 (c) For all  $A$ -modules  $F$ ,  $\text{Tor}_1(E, F) = 0$ .

*Proof:* (a)  $\Rightarrow$  (b). If  $\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$  is a free resolution of  $F$ , then the sequence

$$\dots \rightarrow E \otimes L_n \rightarrow E \otimes L_{n-1} \rightarrow \dots \rightarrow E \otimes L_1 \rightarrow E \otimes L_0 \rightarrow E \otimes F \rightarrow 0$$

is exact, thus  $\text{Tor}_n(E, F) = 0$  for all  $n \geq 1$ .

(b)  $\Rightarrow$  (c) clear. (c)  $\Rightarrow$  (a): If the sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  is exact, so is also (by (3) above)  $\text{Tor}_1(E, F'') \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ . Now  $\text{Tor}_1(E, F'') = 0$ , thus  $E$  is flat.

*Proposition 4:* If  $I$  and  $J$  are two ideals in  $A$ , then  $\text{Tor}_1^A(A/I, A/J) = I \cap J / I \cdot J$ .

*Proof:* From the exact sequence  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ , we get the exact sequence:

$$\text{Tor}_1(A, A/J) \rightarrow \text{Tor}_1(A/I, A/J) \rightarrow I \otimes A/J \rightarrow A \otimes A/J \rightarrow A/I \otimes A/J \rightarrow 0.$$

But now  $\text{Tor}_1(A, A/J) = 0$  ( $A$  being  $A$ -free), and  $I \otimes A/J = I/I \cdot J$ ;  $A \otimes A/J = A/J$ . Therefore the sequence  $0 \rightarrow \text{Tor}_1(A/I, A/J) \rightarrow I/I \cdot J \rightarrow A/J$  is exact, and  $\text{Tor}_1(A/I, A/J) = \text{Ker}(I/I \cdot J \rightarrow A/J) = I \cap J / I \cdot J$ .

*Example :* Let  $U$  be an open set in  $\mathbf{C}^n$ , and  $x \in U$ . Further let  $X, Y \subset U$  be two hypersurfaces, defined by  $I = (f)$  and  $J = (g)$ . Supposing that  $f$  and  $g$  do not have common factors:  $I_x \cap J_x = I_x J_x$ , and

$$\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = \text{Tor}_1(\mathcal{O}_{U,x}/I_x, \mathcal{O}_{U,x}/J_x) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0.$$

*Heuristic remark :* The formula  $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$  expresses the fact that  $X$  and  $Y$  are “in general position”. If for example  $X$  and  $Y$  are two linear subspaces in  $\mathbf{C}^n$  of dimensions  $p$  and  $q$ , we have  $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$  if  $\dim(X \cap Y) = p + q - n$ , and  $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$  otherwise.

Next we shall prove an elementary flatness criterion.

*Proposition 5 :* Let  $E$  be an  $A$ -module. The following conditions are equivalent:

- (a)  $E$  is flat.
- (b) For all finitely generated ideals  $I$  of  $A$ ,  $\text{Tor}_1(E, A/I) = 0$ .
- (c) For all monogenous  $A$ -modules  $F$ ,  $\text{Tor}_1(E, F) = 0$ .

*Proof :* (a)  $\Rightarrow$  (b), by prop. 3.

(b)  $\Rightarrow$  (c): Because Tor is compatible with inductive limit, we can suppose, that  $\text{Tor}_1(E, A/I) = 0$  for an arbitrary ideal  $I$  of  $A$ . But every monogenous  $A$ -module  $F$  can be represented by  $A/I$ .

(c)  $\Rightarrow$  (a). By prop. 3 it is sufficient to prove that  $\text{Tor}_1(E, F) = 0$  for any  $A$ -module  $F$ .

First consider the case, where  $F$  is finitely generated. We use induction, supposing that  $\text{Tor}_1(E, F) = 0$ , when  $F$  has  $n$  generators. Let  $F$  have  $(n+1)$  generators  $x_1, \dots, x_n, x_{n+1}$ . If  $F'$  is the submodule generated by  $\{x_1, \dots, x_n\}$ , then  $F' \subset F$  and  $F'' = F/F'$  is monogenous. The exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  gives the exact sequence  $\text{Tor}_1(E, F') \rightarrow \text{Tor}_1(E, F) \rightarrow \text{Tor}_1(E, F'')$ . Now  $\text{Tor}_1(E, F') = \text{Tor}_1(E, F'') = 0$ , thus  $\text{Tor}_1(E, F) = 0$ . In the general case,  $F$  can be considered as an inductive limit of finitely generated modules, and because Tor is compatible with inductive limits,  $\text{Tor}_1(E, F) = 0$ .

*Proposition 6 :* Let  $A$  be an integral domain, and  $E$  an  $A$ -module. Then  $E$  is torsionfree if and only if  $\text{Tor}_1(E, A/(a)) = 0$ , for any element  $a \in A$ .

*Proof :* If  $E$  is  $A$ -module,  $a \in A$ , then the exact sequence  $0 \rightarrow A \xrightarrow{aI} A \rightarrow A/(a) \rightarrow 0$  gives the exact sequence  $0 \rightarrow \text{Tor}_1(E, A/(a)) \rightarrow E \xrightarrow{aI} E$ . In other words  $\text{Tor}_1(E, A/(a)) = \{x \in E \mid ax = 0\}$ , from which the result follows.

*Corollary*: Let  $A$  be a principal ideal domain.  $E$  is flat if and only if  $E$  is torsionfree.

*Proof*: We have already proved that, if  $E$  is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

*Theorem 2*: Let  $A$  be a noetherian local ring with maximal ideal  $m$ ;  $k = A/m$ , and  $E$  a finitely generated  $A$ -module. The following conditions are equivalent:

- (a)  $E$  is free.
- (b)  $E$  is flat.
- (c)  $\text{Tor}_1^A(E, k) = 0$ .

*Proof*: We have already proved  $(a) \Rightarrow (b) \Rightarrow (c)$ .

$(c) \Rightarrow (a)$ : We recall first Nakayma's lemma. If  $A$  is a local ring with maximal ideal  $m$ ;  $k = A/m$ , and  $E$  is a finitely generated  $A$ -module, such that  $k \otimes_A E = E/mE = 0$ , then  $E = 0$ .

The module  $\bar{E} = k \otimes_A E = E/mE$  is a finitely generated vector space over  $k$ . Let  $\{\bar{x}_1, \dots, \bar{x}_r\}$  be a base of  $\bar{E}$  (over  $k$ ), and  $\{x_1, \dots, x_r\}$   $E$  representatives of  $\bar{x}_i$ :  $s$ . Consider the homomorphism  $\phi : A^r \rightarrow E$ ,  $\phi(a_1, \dots, a_r) = \sum a_i x_i$ . Denoting by  $R$  and  $Q$  the kernel and the cokernel of  $\phi$ , we get an exact sequence:

$$(*) \quad 0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow Q \rightarrow 0$$

and  $R, Q$  are finitely generated  $A$ -modules. From  $(*)$  we get the exact sequence

$$A^r \otimes_A k \rightarrow E \otimes_A k \rightarrow Q \otimes_A k \rightarrow 0.$$

But  $\bar{E} = E \otimes_A k \simeq k^r = A^r \otimes_A k$ , so  $Q \otimes_A k = 0$ , and by Nakayama's lemma  $Q = 0$ .

Therefore we have an exact sequence

$$0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow 0.$$

From this we get:  $\text{Tor}_1(E, k) \rightarrow k \otimes_A R \rightarrow k^r \rightarrow \bar{E} \rightarrow 0$  (exact). Now:  $\bar{E} \simeq k^r$ ,  $\text{Tor}_1(E, k) = 0$  (by assumption). Therefore  $k \otimes_A R = 0$ , and once more by Nakayama's lemma  $R = 0$ , thus  $E \simeq A^r$ , i.e.  $E$  is free.

*Proposition 7:* Let  $\phi : A \rightarrow B$  be a ring homomorphism, and let  $B$  be  $A$ -flat. If  $I$  is an ideal of  $A$ , we write  $\bar{A} = A/I$ ,  $\bar{B} = B/IB = \bar{A} \otimes_A B$ . Let  $F$  be a  $B$ -module, then:  $\text{Tor}_i^A(\bar{A}, F) = \text{Tor}_i^B(\bar{B}, F)$  ( $i \geq 0$ ).

*Proof:* We choose first a  $B$ -free resolution of  $F$

$$\rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0.$$

If  $L.$  is the respective complex of resolution, then

$$\bar{B} \otimes_B L. = B/IB \otimes_B L. = \bar{A} \otimes_A (B \otimes_B L.) = \bar{A} \otimes_A L.$$

Because every  $L_i$  is  $B$ -free, and  $B$  is  $A$ -flat, every  $L_i$  is  $A$ -flat (Property 3 after Th. 1). Thus  $L.$  is a flat  $A$ -resolution, and

$$\text{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \otimes_A L.) = H_i(\bar{B} \otimes_B L.) = \text{Tor}_i^B(\bar{B}, F).$$

We shall next state the second flatness criterion for noetherian local rings.

*Theorem 3:* Let  $A$  and  $B$  be two noetherian local rings, with maximal ideals  $\underline{m}, \underline{n}; k = A/\underline{m}$ . If  $\phi : A \rightarrow B$  is a local homomorphism (i.e.  $\phi(\underline{m}) \subset \underline{n}$ ), and  $F$  finitely generated  $B$  module then

$$F \text{ is } A\text{-flat} \Leftrightarrow \text{Tor}_1^A(k, F) = 0.$$

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, (i)  $\Leftrightarrow$  (iii), p. 98.

The conditions in Bourbaki's theorem are here fulfilled:

- 1° A finitely generated module  $F$  over a noetherian local ring  $B$  is idealwise separated for  $\underline{n}$ . (*Ibid.*, § 5. 1. Ex. 1, p. 97.)
- 2° If  $\phi : A \rightarrow B$  is a local homomorphism,  $F$  is also idealwise separated for  $\underline{m}$ . (*Ibid.*, § 5, prop. 2, p. 101.)
- 3° Also the flatness condition is fulfilled, because  $k$  is a field.

*Remark:* The main interest of the theorem lies in the fact, that it is true without any assumption of finiteness on  $B$ .

*Corollary:* If the assumptions are the same as in the theorem 3, and if moreover  $B$  is  $A$ -flat, then

$$F \text{ is } A\text{-flat} \Leftrightarrow \text{Tor}_1^B(\bar{B}, F) = 0,$$

where  $\bar{B} = B/\underline{m}B$ .



*Proof:*  $\text{Tor}_1^A(k, F) = \text{Tor}_1^B(\bar{B}, F)$ , by prop. 7.

§ 5. *Geometric applications of the flatness criterions*

A) *Flatness for finite morphisms*

*Proposition 1:* Let  $\pi: X \rightarrow S$  be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then  $\pi_*(\mathcal{O}_X)$  is a coherent analytic sheaf over  $S$ . The following conditions are equivalent:

- (a)  $\pi$  is flat (i.e. for every  $x \in X$ ,  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module,  $s = \pi(x)$ ).
- (b) For every  $s$ ,  $(\pi_* \mathcal{O}_X)_s$  is a flat  $\mathcal{O}_{S,s}$ -module.
- (c)  $\pi_* \mathcal{O}_X$  is a locally free sheaf.

*Proof:* Because  $\pi$  is finite  $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$ , thus the only point to prove is (b)  $\Rightarrow$  (c).

Now if  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module, then (by theorem 2)  $\mathcal{O}_{X,x}$  is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

*Proposition 2:* Let  $S$  be a reduced analytic space and  $\mathcal{E}$  a coherent  $\mathcal{O}_S$ -module. Let  $E(s)$  be the finite dimensional vector space (over  $\mathbb{C}$ )  $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C}_s$ .  $\mathcal{E}$  is a locally free  $\mathcal{O}_{S,s}$ -module if and only if  $\dim_{\mathbb{C}} E(s)$  is locally constant.

*Proof:* If  $\mathcal{E}$  is locally free, then  $\dim_{\mathbb{C}} E(s)$  is locally constant. Suppose now that  $\dim_{\mathbb{C}} E(s)$  is locally constant in an open set  $U \subset S$ , and that  $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \rightarrow \mathcal{E}_U \rightarrow 0$  is exact.  $d$  is determined by a  $p \times q$  matrix of analytic functions on  $U$ , so it gives a morphism  $\mathbb{C}_U^p \xrightarrow{d} \mathbb{C}_U^q$  of trivial vector bundles over  $U$ .

From the exact sequence  $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \rightarrow \mathcal{E}_s \rightarrow 0$ , we get (by making tensor-products with  $\mathbb{C}_s$ ) the exact sequence:

$$\mathbb{C}_s^p \xrightarrow{d(s)} \mathbb{C}_s^q \rightarrow E(s) \rightarrow 0,$$

which shows that  $d$  has constant rank in  $U$ . Thus  $\text{Ker } d$  and  $\text{Im } d$  are vector bundles, and we can write

$$\mathbb{C}_U^p = F_1 \oplus G_1, \quad \mathbb{C}_U^q = F_0 \oplus G_0,$$

$$d : \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_0. \end{cases}$$

Now  $\mathcal{E} \simeq$  the sheaf of analytic sections of  $G_0$ , therefore  $\mathcal{E}$  is locally free.

*Definition 1:* Let  $\pi : X \rightarrow S$  be a finite morphism of analytic spaces, and  $s \in S$ . For each  $x \in X(s) = \pi^{-1}(s)$ ,  $\mathcal{O}_{X(s),x} = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$  is finite dimensional vectorspace over  $\mathbf{C}$ . Denote its dimension by  $v(x)$ . Then the degree  $v(s)$  of  $s$  is defined by  $v(s) = \sum_{x \in X(s)} v(x)$ .

*Theorem 1:* Let  $\pi : X \rightarrow S$  be a finite morphism of analytic space and let  $S$  be a reduced space. Then  $X$  is flat over  $S$  if and only if  $v(s)$  is locally constant function of  $s$ .

$$\begin{aligned} \text{Proof: } v(s) &= \sum_{x \in X(s)} \dim_{\mathbf{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbf{C}} \left( \bigoplus_{x \in X(s)} \mathcal{O}_{X(s),x} \right) \\ &= \dim_{\mathbf{C}} \left( \bigoplus_{x \in X(s)} (\mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}) \right) \\ &= \dim_{\mathbf{C}} \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \pi_* (\mathcal{O}_X)_s = \dim_{\mathbf{C}} E(s). \end{aligned}$$

The theorem follows from propositions 1 and 2.

### Examples of flat morphisms

*Example 1:* If  $\pi : X \rightarrow S$  is a local isomorphism near  $x$ , then  $\pi$  is flat at  $x$ .

*Example 2:* Consider § 2, Ex. 1. Here  $v(x) = 1$ .

### Examples of non-flat morphisms

*Examples 1:* If  $X \subset S$  is a closed subspace, not open,  $v(s)$  is not locally constant.

*Example 2:* Let  $X$  be a subspace of  $\mathbf{C}^4$  defined by the ideal intersection of  $(x_3, x_4)$  and  $(x_1 - x_1, x_4 - x_2)$  (which is equal to the product ideal) and let  $\pi$  be the projection onto the  $(x_1, x_2)$ -plane  $\mathbf{C}^2$ . Then  $X$  is a union of two 2-planes in  $\mathbf{C}^4$ , whose intersection is  $(0)$ . When  $s \neq 0$ ,  $X(s)$  consists of two simple points, so  $v(s) = 2$ .  $X(0)$  is given by the ideal  $(x_1, x_2, x_3^2, x_3x_4, x_4^2)$ , thus  $v(0) = 3$ .

*Example 3:* Let  $S = \{(u, v, w) \in \mathbf{C}^3 \mid v^2 = uw\}$  and  $\pi : \mathbf{C}^2 \rightarrow S$  be the map  $(x, y) \rightarrow (x^2, xy, y^2)$ . This map identifies  $S$  with the quotient of  $\mathbf{C}^2$  by the equivalence relation identifying  $(x, y)$  with  $(-x, -y)$ . However,  $\pi$  is not flat, since for  $s \in S$ ,  $v(s) = 2$  if  $s \neq 0$  and  $v(s) = 3$  if  $s = 0$ .

B) *Projection of a product of analytic spaces*

*Theorem 2:* Let  $S$  and  $X$  be analytic spaces. If  $\pi : S \times X \rightarrow S$  is the projection morphism, then  $\pi$  is flat, i.e.  $\mathcal{O}_{S \times X, (s,x)}$  is a flat  $\mathcal{O}_{S,s}$  module for every  $(s, x) \in S \times X$ .

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when  $S$  is a manifold, and finally in the general case.

(a) *Koszul complex*

Let  $A$  be a ring,  $M$  an  $A$ -module and  $h_1, \dots, h_n$  homomorphisms  $M \rightarrow M$ , which commute with each other, i.e.  $h_i h_j = h_j h_i$  for every  $i, j$ .

If  $1 \leq k \leq n$ , set  $Q_k = M/h_1(M) + \dots + h_k(M)$ , and  $Q_0 = M$ , thus, in particular,  $Q_n = Q = M / \sum_{i=1}^n h_i(M)$ . Every  $h_k$  induces a map  $\tilde{h}_k : Q_{k-1} \rightarrow Q_{k-1}$ .

*Definition 2:* The sequence  $(h_1, \dots, h_n)$  is called regular if each of the mappings  $\tilde{h}_k$  ( $1 \leq k \leq n$ ) is injective.

The Koszul complex of the module  $M$  and of the mappings  $h_k$  ( $1 \leq k \leq n$ )  $K. = K. [M; h_1, \dots, h_n]$  is defined in the following way:

$$K_i = \wedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \quad 0 \leq i \leq n.$$

We define the homomorphisms  $d_i : K_i \rightarrow K_{i-1}$  ( $i > 0$ ) by  $\lambda \otimes x \rightarrow \sum_i (e_i \wedge \lambda) \otimes \otimes h_i(x)$ , where  $(e_i)$  is the natural base of  $A^n$ . We also define  $\varepsilon : K_0 \rightarrow Q$  as the natural map  $: K_0 = M \rightarrow M / \sum_{i=1}^n h_i(M) = Q$ . Using the fact that  $h_1, \dots, h_n$  commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also  $\varepsilon d_1 = 0$ . Thus  $K.$  is really a complex.

*Theorem 3 (Poincaré-Koszul).*

If  $(h_1, \dots, h_n)$  is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

If  $h_i \in A$ , it defines the map:  $A \xrightarrow{h_i I} A$ , which we denote also by  $h_i$ . We say that  $(h_1, \dots, h_n)$  is a regular sequence of elements if  $(h_1 I, \dots, h_n I)$  is a regular sequence.

*Corollary.* If  $(h_1, \dots, h_n)$  is a regular sequence of elements, then the Koszul complex  $K. = K. [A; h_1, \dots, h_n] = \{ \wedge^{n-1} A^n \simeq A^{(n)} \}$  is a free resolution of  $Q = A/(h_i)$  ( $(h_i)$  is the ideal generated by  $h_1, \dots, h_n$ ).

*Example:* If  $A = \mathbf{C} \{x_1, \dots, x_n\}$ ;  $h_i = x_i$ , then  $Q_k = A/(x_1, \dots, x_k) = \mathbf{C} \{x_{k+1}, \dots, x_n\}$  and  $Q = Q_n = \mathbf{C}$ . The complex  $K. = K. [A; x_1, \dots, x_n]$  is a free resolution of  $\mathbf{C}$ .

(b) *Proof of theorem 2, when  $S$  is a complex manifold*

In this case we can take  $\mathcal{O}_{S,S} = \mathbf{C} \{t_1, \dots, t_m\} = A$  and if  $\mathcal{O}_{X,x} = \mathbf{C} \{x_1, \dots, x_n\}/(f_1, \dots, f_p)$ , then

$$\mathcal{O}_{S \times X, (s,x)} = \mathbf{C} \{t_1, \dots, t_m, x_1, \dots, x_n\}/(f_1, \dots, f_p) = B.$$

$B$  is an  $A$ -module in a natural way.

By the corollary of the Poincaré-Koszul theorem  $K. = K. [A; t_1, \dots, t_m]$  in a free resolution of  $\mathbf{C}$ . We want to compute the modules  $\text{Tor}_i^A(\mathbf{C}, B) = H_i(K. \otimes B)$  ( $i > 0$ ).

It's easily seen, that we can consider the complex  $K. \otimes B$  as a Koszul

complex  $K'. = K. [B; t_1, \dots, t_m]$  (where  $t_i : B \xrightarrow{t_i I} B$ ). But now the sequence  $(t_1, \dots, t_m)$  is regular, thus by the Poincaré-Koszul theorem  $H_i[K'] = 0$  if  $i > 0$ .

In particular:  $\text{Tor}_1^A(\mathbf{C}, B) = H_1[K. \otimes B] = H_1[K'] = 0$ . By the second flatness criterion  $B$  is  $A$ -flat.

(c) *The general case*

The question being local, we can suppose that  $S \subset W \subset \mathbf{C}^n$ , where  $W$  is open, and  $S$  an analytic subspace of  $W$ . Let  $S$  be defined by  $g_1, \dots, g_r$ . Then  $S \times X \subset W \times X$  and  $\mathcal{O}_S = \mathcal{O}_W/(g_1, \dots, g_r)$ . On the other hand  $\mathcal{O}_{S \times X} = \mathcal{O}_{W \times X}/(g_1, \dots, g_r) = \mathcal{O}_S \otimes_{\mathcal{O}_W} \mathcal{O}_{W \times X}$ . The last equality follows from

the fact, that if  $\pi : X \rightarrow S$  is a morphism, and  $S' \subset S$  a subspace,  $X' = \pi^{-1}(S')$ ,

$$\text{then } \mathcal{O}_{X'} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X.$$

*Remark* : This a particular case of the following proposition: if  $\pi$  and  $\pi'$  are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that  $\mathcal{O}_{W \times X}$  is  $\mathcal{O}_W$ -flat, so by scalar extension  $\mathcal{O}_{S \times X}$  is  $\mathcal{O}_S$  flat.

*Corollary* : If  $X$  and  $S$  are two manifolds and  $\pi : X \rightarrow S$  is a submersion, then  $\pi$  is flat.

### III. PRIVILEGED POLYCYLINDERS

#### § 1. Banach vector bundles over an analytic space

Let  $E$  be a Banach space and  $X$  an analytic space. We denote then by  $E_X$  the trivial bundle  $X \times E$  over  $X$ .

To define bundle morphisms, we first define the sheaf  $\mathcal{H}_X(E)$  of germs of analytic morphisms from  $X$  to  $E$ . If  $U \subset \mathbb{C}^n$  is open, then the set  $\mathcal{H}(U, E)$  of analytic morphisms from  $U$  into  $E$  consists of all functions  $g : U \rightarrow E$  having at every point  $x \in U$  a converging power series expansion.

Let now  $X'$  be a local model for  $X$ , i.e.  $X'$  is the support of the quotient sheaf  $\mathcal{O}_U/J$ , where  $U \subset \mathbb{C}^n$  is open and  $J$  is a coherent sheaf of ideals of  $\mathcal{O}_U$ , then  $\mathcal{H}_{X'}(E)$  is the sheaf associated to the presheaf  $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$  ( $V \subset U$ ,  $V$ -open).

*Remark* : If  $X'$  is reduced, the sections of  $\mathcal{H}_{X'}(E)$  are just the functions from  $X'$  to  $E$  which are locally induced by analytic functions on open sets in  $U$ .

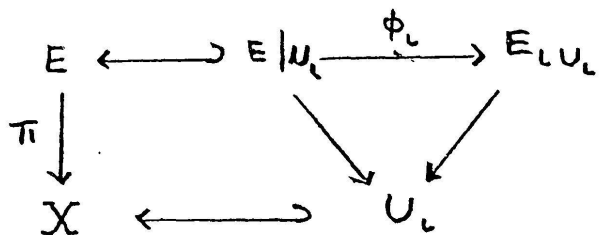
The sheaf  $\mathcal{H}_X(E)$  is constructed with help of the local models  $X'$  of  $X$ , i.e.  $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$ , for every local model  $X'$ .

*Definition 1* : The set of analytic morphisms from an analytic space  $X$  into a Banach space  $E$  is the set  $\mathcal{H}(X; E)$  of sections of the sheaf  $\mathcal{H}_X(E)$ .

Let  $\mathcal{L}(E, F)$  be the Banach space of all continuous linear mappings from the Banach space  $E$  into the Banach space  $F$ .

*Definition 2* : An analytic vector bundle morphism from  $E_X$  into  $F_X$  is an analytic morphism from  $X$  into  $\mathcal{L}(E, F)$ .

Let  $E$  be a topological space,  $X$  an analytic space, and  $\pi : E \rightarrow X$  a continuous projection.



Suppose that  $X$  has an open covering  $(U_\iota)_{\iota \in I}$ , and that for every  $\iota \in I$  there is given a trivial Banach space bundle  $E|_{U_\iota}$  and a homeomorphism  $\phi_\iota$ , such that the following diagram is commutative:

We suppose further that for each pair  $\iota, \kappa \in I$  there is given an analytic vector bundle morphism  $\gamma_{\iota\kappa} : E|_{U_\iota \cap U_\kappa} \rightarrow E|_{U_\iota \cap U_\kappa}$ , with the underlying mapping  $\phi_\iota \circ \phi_\kappa^{-1}$ , such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \quad \text{for all } \iota, \kappa, \lambda \in I.$$

This data gives a Banach vector bundle atlas on  $E$  and provides  $E$  with the structure of a Banach vector bundle over  $X$  (two atlases are equivalent if there exists an atlas containing both).

*Remark:* If  $X$  is reduced, the  $\gamma_{\iota\kappa}$  are determined by their underlying map and the condition  $\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}$  is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

*Proposition 1:* Let  $\phi : E \rightarrow F$  be a morphism of two Banach vector



bundles  $E$  and  $F$ , and  $x \in X$ .

If  $\phi_x \in \mathcal{L}(E(x), F(x))$  is an isomorphism, then there exists an open neighbourhood  $U \subset X$  of  $x$ , such that  $\phi|_U : E|_U \rightarrow F|_U$  is a vector bundle isomorphism.

*Proof:* First we take a trivialisation  $E|_V = E_0|_V, F|_V = F_0|_V$  at  $x \in V \subset X$  ( $V$ -open).

The set  $\text{Isom}(E_0, F_0)$  of isomorphic mappings is an open subset of  $\mathcal{L}(E_0, F_0)$  and the mapping  $g \rightarrow g^{-1}$  is an analytic isomorphism:

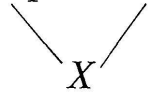
$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0).$$

So we have in an open neighbourhood  $U \subset X$  of  $x$  an analytic morphism  $y \rightarrow \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$ , which defines the inverse morphism  $(\phi|_U)^{-1} : F|_U \rightarrow E|_U$ .

*Definition 3 :* Let  $E$  and  $F$  be two Banach spaces and  $f$  a continuous linear mapping from  $E$  into  $F$ .  $f$  is a *split mono-(epi) morphism*, if there exists a mapping  $g \in \mathcal{L}(F, E)$  such that  $g \circ f = I_E$ . (Resp.  $f \circ g = I_F$ .)

*Definirion 4 :* Let  $E_1$  and  $E_2$  be two Banach vector bundles over an analytic space  $X$ , and  $f$  a vector bundle morphism from  $E_1$  into  $E_2$ .  $f$  is a *split mono (epi) morphism*, if there exists a vector bundle morphism  $g : E_2 \rightarrow E_1$  such that  $g \circ f = I_{E_1}$ . (Resp.  $f \circ g = I_{E_2}$ .)

Equivalently,  $f : E_1 \rightarrow E_2$  is a split monomorphism if and only if  $E_2$  can



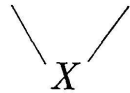
be decomposed in a direct sum  $E_2 = F_2 \oplus G_2$  such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \rightarrow G_2 \end{cases}.$$

and  $f$  is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that } f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq E_2 \end{cases}.$$

*Proposition 2 :* Let  $E \xrightarrow{\phi} F$  be a bundle morphism and  $x \in X$ .



If  $\phi_x : E(x) \rightarrow F(x)$  is a split epi (mono) morphism, then the point  $x$  has an open neighbourhood  $U \subset X$ , such that  $\phi|_U : E|_U \rightarrow F|_U$  is a split vector bundle epi (mono) morphism.

*Proof :* Suppose that  $\phi_x$  is a split epimorphism. We take first a trivialisaton  $E|_V = E_{0V}, F|_V = F_{0V}$  at  $x$ , so that there exists a mapping  $\sigma \in \mathcal{L}(F_0, E_0)$ ,  $\phi_x \circ \sigma = I_{F_0}$ . If we define a morphism  $\psi : F_{0V} \rightarrow E_{0V}$  by  $x \rightarrow \sigma \in \mathcal{L}(F_0, E_0)$ , the morphism  $\gamma = \phi \circ \psi : F_{0V} \rightarrow E_{0V}$  has an isomorphic fibre mapping  $\gamma_x = I_{F_0}$  in  $x$ . By proposition 1 we have an isomorphic restriction  $\gamma|_U, \phi|_U \circ (\psi|_U \circ (\gamma|_U)^{-1}) = I_{F_{0U}}$ .

When  $\phi_x$  is a split monomorphism, the proof is similar.

*Definition 5 :* Let  $B_1, B_2, B_3$  be Banach spaces, and  $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$  continuous linear mappings. This sequence forms a *complex*, if  $k \circ j = 0$ . This sequence is *split exact* if the space  $B_i$  can be decomposed in direct

sums  $B_i = C_i \oplus D_i$  such that

$$j: \begin{cases} C_1 \rightarrow 0 \\ D_1 \simeq C_2 \end{cases} \quad k: \begin{cases} C_2 \rightarrow 0 \\ D_2 \simeq C_3 \end{cases} .$$

*Definition 6:* A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \\ & & X & & \end{array} \quad \text{is a complex if } g \circ f = 0.$$

The sequence is *split exact*, if every  $E_i$  can be decomposed  $E_i = F_i \oplus G_i$ , such that:

$$f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_2 \end{cases} \quad g: \begin{cases} F_2 \rightarrow 0 \\ G_2 \simeq F_3 \end{cases} .$$

*Theorem 1:* Let  $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \end{array}$  be a complex of Banach vector

bundles and  $x_0 \in X$ .

If the sequence of Banach spaces  $E_1(x_0) \xrightarrow{f_{x_0}} E_2(x_0) \xrightarrow{g_{x_0}} E_3(x_0)$  is split exact, then there exists an open neighbourhood  $U \subset X$  of  $x_0$ , such that  $E_1|_U \xrightarrow{f|_U} E_2|_U \xrightarrow{g|_U} E_3|_U$  is a split exact sequence of Banach vector bundles.

*Proof:* We take a neighbourhood  $V$  of  $x$ , such that we have a complex  $E_{1V} \xrightarrow{f|_V} E_{2V} \xrightarrow{g|_V} E_{3V}$  of trivial bundles. By assumption we have the decompositions  $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$  with

$$f_{x_0}: \begin{cases} F_1(x_0) \rightarrow 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \quad g_{x_0}: \begin{cases} F_2(x_0) \rightarrow 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2,  $f|_V : G_{1V} \rightarrow E_{2V}$ ,  $g|_V : G_{2V} \rightarrow E_{3V}$  are both split monomorphisms in a neighbourhood  $W \subset V$  of  $x_0$  and the images  $F_2 = f(G_{1W})$ ,  $F_3 = g(G_{2W})$  are subbundles of  $E_{2W}$  esp.  $E_{3W}$ , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W} .$$

By our construction



$$g|_W : \begin{cases} F_2 & \rightarrow 0 \\ G_2 W & \simeq F_3 \end{cases} .$$

If  $p: E_{2W} \rightarrow F_2$  is the projection with kernel  $G_{2W}$ , the map,  $p \circ f: E_{1W} \rightarrow F_2$  is a split epimorphism in  $x_0$ . Again by prop. 2 we have over an open neighbourhood  $U \subset W$  of  $x_0$  a decomposition  $E_{1U} = F_1 \oplus G_{1U}$  (with  $F_1 = \text{Ker } p \circ f$ )

$$(p \circ f)|_U : \begin{cases} F_1 & \rightarrow 0 \\ G_{1U} & \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image  $f|_U(F_1)$  is contained in  $G_{2U}$ . But  $g|_U \circ f|_U = 0$  and  $g|_{G_{2U}}$  is a monomorphism hence  $f|_U: F_1 \rightarrow 0$ . We get finally (restricting all our morphisms to  $U$ )

$$f|_U : \begin{cases} F_{1U} & \rightarrow 0 \\ G_{1U} & \simeq F_{2U} \end{cases} \qquad g|_U : \begin{cases} F_{2U} & \rightarrow 0 \\ G_{2U} & \xrightarrow{\sim} F_{3U} \end{cases} .$$

## § 2. Privileged polycylinders

*Definition 1:* A polycylinder in  $\mathbf{C}^n$  is a compact set  $K$  of the form  $K = K_1 \times \dots \times K_n$  where each  $K_i$  is a compact, convex subset of  $\mathbf{C}$ , with nonempty interior. If each  $K_i$  is a disc, then  $K$  is a polydisc. We first recall the following theorem of Cartan.

*Theorem 1:* Let  $K$  be a polycylinder contained in an open subset  $U$  of  $\mathbf{C}^n$ . Let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ .

(A) There exists an open neighbourhood of  $K$  over which  $\mathcal{F}$  admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

(B)  $H^q(K, \mathcal{F}) = 0$  for  $q > 0$ .

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf  $\mathcal{F}$ , the sequence

$$0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0$$

is an  $\mathcal{O}_U(K)$  - free resolution of  $\mathcal{F}(K)$ .

2) Given a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0,$$

then the sequence

$$0 \rightarrow \mathcal{F}'(K) \rightarrow \mathcal{F}(K) \rightarrow \mathcal{F}''(K) \rightarrow 0 \quad \text{is exact.}$$

Let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ , and let  $K \subset U$  be a polycylinder. If  $V$  is an open neighbourhood of  $K$ , then  $\mathcal{F}(V)$  can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give  $\mathcal{F}(K)$  the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from  $\mathcal{F}(K)$  and by choosing  $K$  in a "privileged" way.

Let  $B(K) = \{f : K \rightarrow \mathbb{C} \mid f \text{ continuous on } K \text{ and analytic on } \overset{\circ}{K}\}$ , then  $B(K)$  is Banach algebra and  $B(K) \subset C(K)$ . The sections of  $\mathcal{O}_U$  over  $K$  are elements of  $B(K)$ , and  $B(K)$  is in fact the uniform closure of  $\mathcal{O}_U(K)$  in  $C(K)$ .

If  $\mathcal{L} = \mathcal{O}_U^r$ , we define  $B(K, \mathcal{L}) = B(K)^r$ . Then  $B(K; \mathcal{L})$  is a free  $B(K)$ -module, and since  $\mathcal{L}(K) = \mathcal{O}_U(K)^r$ , we have  $B(K; \mathcal{L}) = B(K) \otimes_{\mathcal{O}_U(K)} \mathcal{L}(K)$ .

We now assume that  $\mathcal{F}$  is a coherent sheaf on  $U$ , where  $U \subset \mathbb{C}^n$  is open. Consider a free resolution

$$(R) \quad 0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 \quad \text{of } \mathcal{F}.$$

From (R) we get an  $\mathcal{O}_U(K)$ -free resolution of  $\mathcal{F}(K)$

$$(R') \quad 0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_1(K) \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0.$$

Taking the tensorproduct  $B(K) \otimes_{\mathcal{O}_U(K)}$  we get the complex

$$B(K; \mathcal{L}.) : 0 \rightarrow B(K; \mathcal{L}_n) \rightarrow \dots \rightarrow B(K; \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0).$$

*Definition 2:* The polycylinder  $K$  is called  $\mathcal{F}$ -privileged if the complex  $B(K; \mathcal{L}.)$  is split-exact in every degree  $> 0$ .

*Remark:* The property of being  $\mathcal{F}$ -privileged is independent of the resolution (R).

The exactness of  $B(K; \mathcal{L}.)$  can be expressed by  $\text{Tor}_i^{\mathcal{O}_U(K)}(B(K), \mathcal{F}(K)) = 0$ , for every  $i > 0$ , and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of  $(R)$ , and this is omitted.

Since  $B(K; \mathcal{L}_i)$  is a Banach space, the image and its complement are thus Banach spaces if  $K$  is  $\mathcal{F}$ -privileged. In this case we define  $B(K; \mathcal{F}) = \text{Coker}(B(K, \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{O}_U} \mathcal{F}(K)$  and we get a  $B(K)$ -module, which is a Banach-space.

*Warning:* In the definition of split-exactness, the subspaces are splitting vector spaces, but they are not splitting  $B(K)$ -modules in general.

We have the following important theorem about the existence of privileged polycylinders:

*Theorem 2:* Let  $U$  be an open subset of  $\mathbf{C}^n$ , and let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ . For any  $x \in U$  there exists a fundamental system of neighbourhoods of  $x$  in  $U$ , which are  $\mathcal{F}$ -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

*Example:* (Curves in  $\mathbf{C}^2$ ) Let  $U \subset \mathbf{C}^2$  be an open connected neighbourhood of the origin, and let  $h: U \rightarrow \mathbf{C}$  be analytic and  $h \neq 0$ .

Let  $X$  be the curve given by  $h$ , that is  $X = h^{-1}(0)$ ,  $\mathcal{O}_X = \mathcal{O}_U / (h)$ . We have an exact sequence  $0 \rightarrow \mathcal{O}_U \xrightarrow{h} \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$ . Consider a polycylinder  $K = K_1 \times K_2 \subset U$ . By definition  $K$  is  $\mathcal{O}_X$ -privileged if and only if  $h: B(K) \rightarrow B(K)$  is a split monomorphism.

Let  $\dot{K}_j$  denote the boundary of  $K_j$ , and define  $\ddot{K} = \dot{K}_1 \times \dot{K}_2$  ( $\ddot{K}$  is called the Šilov Boundary of  $K$ ).

*Proposition 1:* (a) The following conditions are equivalent:

- (i)  $h: B(K) \rightarrow B(K)$  is a monomorphism.
- (i')  $\exists a > 0$  such that  $\|hf\| \geq a\|f\|$ ,  $\forall f \in B(K)$ .
- (ii)  $X \cap \ddot{K} = \emptyset$ .

(b) If  $(K_1 \times K_2) \cap X = \emptyset$ , then  $h$  is a split monomorphism (i.e.  $K$  is  $\mathcal{O}_X$ -privileged).

*Proof:* (a) (i)  $\Leftrightarrow$  (i') is a well known fact from the theory of normed vector spaces.

(ii)  $\Rightarrow$  (i'). Assume  $X \cap \ddot{K} = \emptyset$ . If  $f \in B(K)$ , then it follows from the maximum principle that  $\|f\| = \sup_K |f(x)| = \sup_{\ddot{K}} |f(x)|$ . Since  $h(x) \neq 0$

whenever  $x \in \overset{\circ}{K}$ , we get  $a = \inf_K |h(x)| > 0$ . Hence  $\|hf\| = \sup_K |hf(x)| \geq \geq a \sup_K |f(x)| = a \|f\|$ .

(i')  $\Rightarrow$  (ii). Suppose that  $X \cap \overset{\circ}{K} \neq \emptyset$  and  $x = (x_1, x_2) \in X \cap \overset{\circ}{K}$ . We choose an analytic function  $f_1 : U_1 \rightarrow \mathbf{C}$ , where  $U_1 \supset K_1$ , and  $U_1$  is open, such that  $f_1(x_1) = 1$ ,  $|f_1(z)| < 1$  if  $z \in K_1$ ,  $z \neq x_1$ . Similarly we choose an analytic function  $f_2 : U_2 \rightarrow \mathbf{C}$ , with the same properties. Consider the function  $f \in B(K) : (z_1, z_2) \rightarrow f_1(z_1)f_2(z_2)$ . Since  $h(x) = 0$  it follows that the sequence  $\{hf^n\}$  converges pointwise to 0 in  $K$ .

Applying Dini's theorem we get  $\|hf^n\| \rightarrow 0$ . From the inequality  $a \|f^n\| \leq \|hf^n\|$  we get  $\|f^n\| \rightarrow 0$ , which is a contradiction, because for every  $n : f^n(x) = 1$ .

(b) Use the Weierstrass preparation theorem (extended form).

*Question.* Does the condition (ii) imply that  $h : B(K) \rightarrow B(K)$  is a split monomorphism?

#### IV. FLATNESS AND PRIVILEGE

##### § 1. Morphisms from an analytic space into $B(K)$

Let  $S$  be an analytic space and  $K$  a polycylinder in an open set  $U \subset \mathbf{C}^n$ . We want to construct an  $\mathcal{O}_S$ -algebra homomorphism  $\phi : \mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S; B(K))$ .

(a) Consider first  $S = U' \subset \mathbf{C}^m$ ,  $U'$ -open. If  $h \in \mathcal{O}_{U' \times U}(U' \times U)$  and  $s \in U'$ ,  $x \in K$ , define  $(\phi(h)(s))(x) = h(s, x)$ . Using the Cauchy integral, one can show that  $\phi(h)$  is analytic. On the other hand it's obvious that  $\phi$  is an  $\mathcal{O}_{U'}$ -algebra homomorphism.

(b) Let  $S$  have a special model in the polydisc  $\Delta$  in  $\mathbf{C}^m$ , defined by a sheaf  $\mathcal{I}$  of ideals of  $\mathcal{O}_\Delta$ , and let  $\mathcal{J}$  be generated by  $f_1, \dots, f_p$ ,  $V$ -a polycylinder neighbourhood of  $K$  in  $U$ . By Cartan's theorem  $B$  for a polycylinder,

the sequence  $0 \rightarrow \mathcal{I}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \rightarrow 0$  is exact. If we denote by  $\tilde{\pi}$  the projection  $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K))$ ,  $(f_1, \dots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset \subset \text{Ker } \tilde{\pi}$ . Therefore, because  $\pi$  is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$ , such that the diagram

$$\begin{array}{ccc} \mathcal{O}(\Delta \times V) & \xrightarrow{\phi} & \mathcal{H}(\Delta, B(K)) \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ \mathcal{O}(S \times V) & \xrightarrow{\phi} & \mathcal{H}(S, B(K)) \end{array}$$

is commutative;  $\phi$  is evidently an  $\mathcal{O}_S$ -algebra homomorphism.

§ 2. *The flatness and privilege theorem*

*Notation*

Let  $S$  be an analytic space,  $U$  an open set in  $\mathbb{C}^n$ , and  $\pi : S \times U \rightarrow S$  the first projection.

If  $\mathcal{F}$  is an  $\mathcal{O}_{S \times U}$  module, then for every  $s \in S$  we denote by  $\mathcal{F}(s)$  the  $\mathcal{O}_U$ -module  $i_s^* \mathcal{F}$ , where  $i_s$  is the injective morphism  $x \rightarrow (s, x)$  from  $U$  into  $S \times U$ . If  $x \in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s, x)} / m_s \cdot \mathcal{F}_{(s, x)} \simeq \mathcal{F}_{(s, x)} \otimes_{\mathcal{O}_{S, s}} \mathbb{C}_s.$$

*Theorem 1:* Let  $\mathcal{E}$  be a coherent and  $S$ -flat  $\mathcal{O}_{S \times U}$ -module, and  $K$  a poly-cylinder in  $U$ .

(a) When  $K$  is privileged for  $\mathcal{E}(s_0)$ ,  $s_0$  has a neighbourhood  $V$  such that  $K$  is  $\mathcal{E}(s)$ -privileged for each  $s \in V$ . In other words: the set  $S' = \{s \in S \mid K \text{ is } \mathcal{E}(s)\text{-privileged}\}$  is open in  $S$ .

(b) It is possible to define a Banach vector bundle over  $S'$  whose fibre at any  $s \in S'$  is  $B(K, \mathcal{E}(s))$ .

To prove the theorem we need:

*Lemma 1:* Under the conditions of the theorem, we can, for every  $s \in S$ , find a neighbourhood  $W$  of  $\{s\} \times K$  and a free resolution of finite length

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \xrightarrow{\varepsilon} \mathcal{E} \rightarrow 0 \text{ in } W.$$

*Proof:* Let  $(s, x)$  be a point of  $S \times U$  and  $\mathcal{L}_*^0$  a finite resolution of  $\mathcal{F}(x)$  in a neighbourhood of  $x$  (there exists such one, by the theorem of syzygies). We shall show that there exists a resolution  $\mathcal{L}^*$  of  $\mathcal{F}$  in a neighbourhood of  $(s, x)$  such that  $\mathcal{L}^*(s) = \mathcal{L}_*^0$ ; if  $\mathcal{L}_i^0 = \mathcal{O}_x^{r_i}$  define

$$\mathcal{L}_i = \mathcal{O}_{S \times U}^{r_i} \text{ and } \mathcal{K}_i^0 = \text{Ker } d_i^0 : \mathcal{L}_i^0 \rightarrow \mathcal{L}_{i-1}^0.$$

We shall construct by induction (with respect to  $i$ )  $d_i : \mathcal{L}_1 \rightarrow \mathcal{L}_{i-1}$  in a neighbourhood of  $(s, x)$  such that  $d_i(s) = d_i^0$ , and prove that  $\mathcal{K}_i = \text{Ker } d_i$  is  $S$ -flat and that  $\mathcal{K}_i(s) = \mathcal{K}_i^0$ .

$$\begin{array}{ccc} \mathcal{L}_{i+1} & \xrightarrow{d_{i+1}} & \mathcal{K}_i \\ \downarrow & & \downarrow \\ \mathcal{L}_{i+1}^0 & \xrightarrow{d_{i+1}^0} & \mathcal{K}_i^0 \end{array}$$
 Suppose that we have constructed  $d_i$  and proved the properties for  $\mathcal{K}_i$ . We can construct  $d_{i+1} : \mathcal{L}_{i+1} \rightarrow \mathcal{L}_i$  in a neighbourhood of  $(s, x)$  such that the diagram is commutative.

Nakayama's lemma shows that  $\text{Im } d_{i+1} = \mathcal{K}_i$  at the point  $(s, x)$ , therefore in a neighbourhood of that point.

The exact sequence

$$0 \rightarrow \mathcal{K}_{i+1} \rightarrow \mathcal{L}_{i+1} \rightarrow \mathcal{K}_i \rightarrow 0,$$

where  $\mathcal{K}_i$  and  $\mathcal{L}_{i+1}$  are  $S$ -flat, shows that  $\mathcal{K}_{i+1}$  is  $S$ -flat, and that  $\mathcal{K}_{i+1}(s) = \mathcal{K}_{i+1}^0$ . The first step of the induction is analogous.

*Proof of the theorem:* Let  $s_0 \in S$  and

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_1} \mathcal{L}_0 \rightarrow \mathcal{E}|_W \rightarrow 0$$

be a free  $\mathcal{O}_{S \times U}$  resolution of  $\mathcal{E}$  in a neighbourhood  $W = V_1 \times V_2$  of  $\{s_0\} \times K$ . The sheaf  $\mathcal{E}$  is  $\mathcal{O}_S$ -flat, so for each  $s \in V_1$ , the sequence

$$0 \rightarrow \mathcal{L}_p \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \dots \rightarrow \mathcal{L}_1 \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \mathcal{L}_0 \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \mathcal{E}|_W \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow 0$$

is exact. So the sequence

$$(A) \quad 0 \rightarrow \mathcal{L}_p(s) \xrightarrow{d_p(s)} \dots \xrightarrow{d_1(s)} \mathcal{L}_0(s) \rightarrow \mathcal{E}(s)|_{V_2} \rightarrow 0$$

is exact when  $s \in V_1$ . Now  $\mathcal{L}_i(s) \simeq \mathcal{O}_{V_2}^{r_i}$  ( $0 \leq i \leq p$ ) and every  $d_i(s)$  induces a continuous linear map:

$B(K, \mathcal{L}_i(s)) \rightarrow B(K, \mathcal{L}_{i-1}(s))$ , which we also denote by  $d_i(s)$ . We can consider  $d_i = (d_{ijk})$  as an  $r_i \times r_{i-1}$ -matrix with entries from  $\mathcal{O}_{S \times U}(W)$ .

By § 1 we have a  $\mathcal{O}_S$ -algebra homomorphism

$$\mathcal{O}_{S \times W}(S \times W) \rightarrow \mathcal{H}(S, B(K)).$$

From the matrix  $(d_{ijk})$  we get by this homomorphism a morphism  $\tilde{d}_i$ :

$$V_0 \rightarrow \mathcal{L}(B(K)^{r_i}, B(K)^{r_{i-1}}) = \mathcal{L}(B(K, \mathcal{L}_i(s)), B(K, \mathcal{L}_{i-1}(s))).$$

(Here  $V_0$  is some neighbourhood of  $s_0$ ) such that  $\tilde{d}_i(s) = d_i(s)$  for each  $s \in V_0$ . In other words we have a sequence of Banach vector bundle morphisms

$$(B) \quad 0 \rightarrow B(K, \mathcal{L}_p) \xrightarrow{d_p} \dots \xrightarrow{\tilde{d}_1} B(K, \mathcal{L}_0).$$

Using the fact that  $\mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S, B(K))$  is an  $\mathcal{O}_S$ -algebra homomorphism, it easily follows that (B) is complex of Banach vector bundles over  $S$ .

Now  $K$  is  $\mathcal{E}(s_0)$ -privileged, thus

$$0 \rightarrow B(K, \mathcal{L}_p(s_0)) \xrightarrow{d_p(s_0)} \dots \xrightarrow{d_1(s_0)} B(K, \mathcal{L}_0(s_0))$$

is split exact, so by theorem III.1

$$0 \rightarrow B(K, \mathcal{L}_p)|_V \xrightarrow{\tilde{d}_p|_V} \dots \xrightarrow{\tilde{d}_1|_V} B(K, \mathcal{L}_0)|_V$$

is split exact for some neighbourhood  $V$  of  $s_0$ .

Because  $\tilde{d}_i(s) = d_i(s)$  and the sequence (A) is exact part (a) of the theorem follows.

(b)  $B(K, \mathcal{L}_0)|_V$  splits as the direct sum of  $\text{im } \tilde{d}_1$  and a bundle  $E_V$ , such that  $E_{V,s} \simeq B(K, \mathcal{E}(s))$ , for each  $s \in V$ . We must show that these bundle structures fit together globally.

Suppose therefore that  $V$  is open in  $S'$  and that

$$\begin{aligned} 0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \rightarrow \mathcal{E}|_{V \times V_2} \rightarrow 0 \\ 0 \rightarrow \mathcal{L}'_p \xrightarrow{d'_p} \dots \xrightarrow{d'_2} \mathcal{L}'_1 \xrightarrow{d'_1} \mathcal{L}'_0 \rightarrow \mathcal{E}|_{V \times V_2} \rightarrow 0 \end{aligned}$$

are free resolutions of  $\xi$  over  $V \times V_2$ .

If  $V_1, V_2$  are open polycylinders, we can find an  $\mathcal{O}_{S \times U}$ -homomorphism  $\phi_0 : \mathcal{L}'_0 \rightarrow \mathcal{L}_0$  such that

$$\begin{array}{ccc} \mathcal{L}'_0 & \xrightarrow{\varepsilon'} & \mathcal{E}|_{V \times V_2} \rightarrow 0 \\ \phi_0 \uparrow & & \parallel \\ \mathcal{L}_0 & \xrightarrow{\varepsilon} & \mathcal{E}|_{V \times V_2} \rightarrow 0 \end{array}$$

commutes.  $\phi_0$  determines a bundle morphism  $\tilde{\phi}_0: B(K, \mathcal{L}_0) \rightarrow B(K, \mathcal{L}'_0)$ .  $B(K, \mathcal{L}_0)$  (resp.  $B(K, \mathcal{L}'_0)$ ) splits as  $(\text{im } \tilde{d}_1) \otimes E_V$  [Resp.  $(\text{im } \tilde{d}'_1) \otimes E'_V$ ].

Let  $p'$  be the projection morphism:  $B(K, \mathcal{L}'_0) \rightarrow E'_V$  with kernel  $\text{im } \tilde{d}'_1$ , and put  $\tilde{\phi} = p' \circ \phi_0|_{E_V}$ .

The commutative diagram

$$\begin{array}{ccc}
 B(K, \mathcal{L}_0(s)) & \xrightarrow{\tilde{\phi}_0} & B(K, \mathcal{L}'_0(s)) \\
 \varepsilon \downarrow & \nearrow E_{V,s} & \nwarrow E'_{V,s} \\
 & \xrightarrow{\tilde{\phi}} & \\
 & \nwarrow \varepsilon \simeq \alpha \circ \varepsilon & \nearrow \simeq \alpha' \circ \varepsilon' \\
 B(K, \mathcal{E}(s)) & \xleftarrow{\text{id}} & B(K, \mathcal{E}(s))
 \end{array}$$

and the open mapping theorem shows that  $\tilde{\phi}(s)$  is an isomorphism of Banach spaces for each  $s \in V$ , so  $\tilde{\phi}: E_V \rightarrow E'_V$  is a bundle isomorphism. We also notice that  $\tilde{\phi}$  depends only on the choice of splittings in  $B(K, \mathcal{L}_0)$  and  $B(K, \mathcal{L}'_0)$ , and not on the choice of  $\tilde{\phi}_0$ . This ends the proof of the theorem.

*Remark:* Consider the general situation where  $X$  and  $S$  are analytic spaces, and  $\pi: X \rightarrow S$  is a morphism,  $\mathcal{E}$  an  $\mathcal{O}_X$ -module. To study the local dependence of  $\mathcal{E}$  on  $S$ , one can imbed an open set  $X'$  in  $X$  in the open set  $U \subset \mathbb{C}^n$ . The morphism  $\phi: X' \rightarrow U, \pi: X' \rightarrow S$  determine the imbedding  $\pi \times \phi: X' \rightarrow S \times U$  such that the diagram commutes.  $\mathcal{E}$  can be extended by zero into a sheaf  $\mathcal{E}'$  over  $U \times S$ . Obviously this sheaf  $\mathcal{E}'$  is  $S$ -flat iff  $\mathcal{E}$  is  $S$ -flat.

Therefore theorem 1 makes clear also this general situation.

*Corollary:* If  $\pi: X \rightarrow S$  is a morphism and  $\mathcal{E}$  a coherent  $\mathcal{O}_X$ -module. Then  $\pi|_{\text{Supp}(\mathcal{E})}$  is an open map.

*Proof:* Suppose as above that  $X$  is imbedded in  $S \times U$ , and  $\mathcal{E}$  is extended by zero to  $S \times U$ . Let  $x_0 \in \text{Supp } \mathcal{E}$ , and  $V$  be a neighbourhood of  $x_0$  in  $S \times U$ . Let  $s_0 = \pi(x_0)$  and choose an  $\mathcal{E}(s_0)$ -privileged polycylinder  $K$  in  $U$ , such that  $\{s_0\} \times K \subset V$ , over some neighbourhood  $W$  of  $s_0$ . We have the Banach bundle  $B(K, \mathcal{E}|_{\pi^{-1}(W)})$ , whose fiber over  $s$  is  $B(K, \mathcal{E}(s))$ . Since  $x_0 \in \text{Supp } \mathcal{E}(s_0)$  and  $K$  is a neighbourhood of  $x_0$ ,  $B(K; \mathcal{E}(s_0)) \neq 0$ . As all the fibers are isomorphic, then for all  $s \in U$ ,  $B(K; \mathcal{E}(s)) \neq 0$  and therefore  $\{s\} \times K \cap \text{Supp } \mathcal{E} \neq \emptyset$ , and  $s \in \pi(\text{Supp } \mathcal{E})$ . This proves that  $\pi$  is open.



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