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FLATNESS AND PRIVILEGE

by A. DOUADY

I. FLAT MORPHISMS

§ 1. *Analytic subspaces of an analytic space*

Let Y_1 and Y_2 be closed analytic subspaces of an analytic space X, and let them be defined by the $\mathcal{O}_X$ ideals J_1, J_2.

Definition 1: We say that Y_1 is *analytically included* in Y_2, and we write $Y_1 \subset Y_2$, when $J_1 \supset J_2$.

Remark: The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example: $X = (\mathbf{C}, \mathcal{O}_\mathbf{C})$; $J_1 = (x)$, $J_2 = (x^2)$. The space Y_1 is a simple point, Y_2 is a double point, $Y_1 \not\supset Y_2$, while they have the same underlying set.

Definition 2: The subspace $Y_1 \cup Y_2$ is the smallest subspace of X containing Y_1 and Y_2, and it is defined by $J_1 \cap J_2$. The subspace $Y_1 \cap Y_2$ is the biggest subspace of X contained in both Y_1 and Y_2, and it is defined by $J_1 + J_2$.

Remark: The underlying set of $Y_1 \cup Y_2$ (Resp. $Y_1 \cap Y_2$) is the union (Resp. intersection) of the underlying sets of Y_1 and Y_2. However $\cup$ and $\cap$ of analytic spaces do not satisfy the distributivity laws which hold in set-theory: $(Y_1 \cup Y_2) \cap Y_3$ contains $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$, and therefore their union; similarly $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$. In general the converse inclusions do not hold.

Example: Let $X = \mathbf{C}^2$ and Y_1, Y_2, Z be given by ideals $(x-y)$, $(x+y)$ and (x) respectively.

$(Y_1 \cup Y_2) \cap Z$ is $\{0\}$ provided with $\mathbf{C}\{y\}/(y^2)$, while $(Y_1 \cap Z) \cup (Y_2 \cap Z)$ is the reduced space $\{0\}$. On the other hand: $Y_1 \cap Y_2 \subset Z$, $(Y_1 \cap Y_2) \cup Z = Z$, while $(Y_1 \cup Z) \cap (Y_2 \cup Z)$ is the space defined by the ideal (x^2, xy). Its local ring at the origin is $\mathbf{C}\{x, y\}/(x^2, xy)$ in which x is nilpotent.

Definition 3: Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J, and $f = (f_0, f^1) : X' \to X$ a morphism.

The inverse image of Y by f, $f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J)\,\mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f-fiber over x, and is denoted by $f^{-1}(x)$ or $X'(x)$.

Proposition 1: If $f = (f_0, f^1) : X' \to X$ is a morphism of analytic spaces, and Y is a subspace of X, then $f^{-1}(Y) \simeq Y \underset{X}{\times} X'$.

Proof: Let T be any analytic space, and $g: T \to X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y. Thus $f^{-1}(Y)$ and $X' \underset{X}{\times} X$ are solutions of the same universal problem.

§ 2. *Analytic pull-back*

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \underset{A}{\otimes} F$, where A is a commutative ring and E, F are two A-modules.

(1°) $E \otimes A^n = E^n$ $(n \in N)$

(2°) If the sequence of A-modules $F' \to F \to F'' \to 0$ is exact, then also the sequence $E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}$; $f_{ij} : F_j \to F_i$ is an inductive system, then

$$E \otimes \underset{\longrightarrow}{lim}\, F_i = \underset{\longrightarrow}{lim}\,(E \otimes F_i).$$

On the other hand these properties characterize completely the functor $\otimes$.

Definition 1: Let $f = (f_0\ f^1) : X' \to X$ be a morphism of analytic spaces, and $\mathcal{E}$ an $\mathcal{O}_X$-module. Then $f_0^*\mathcal{E}$ is an $f_0^*\,\mathcal{O}_X$-module and $\mathcal{O}_{X'}$ is also an $f_0^*\,\mathcal{O}_X$-module (by $f^1 : f_0^*\,\mathcal{O}_X \to \mathcal{O}_{X'}$).

The analytic pull-back $f^*\,\mathcal{E}$ of $\mathcal{E}$ by f is defined by scalar extension:

$$f^*\,\mathcal{E} = f_0^*\,\mathcal{E} \underset{f_0^*\,\mathcal{O}_X}{\otimes} \mathcal{O}_{X'}$$

Remark: The inverse image is a particular case of the analytic pull-back.

In fact, if Y is a closed analytic subspace of X and $f: X' \to X$ is a morphism:

$$f^* \mathcal{O}_Y = f_0^*(\mathcal{O}_X/J_Y) \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'}/f^1(J_Y).\, \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that $A/I \underset{A}{\otimes} E \simeq E/IE$).

Elementary properties of the analytic pull-back:

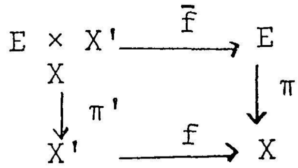
(*a*) $(f^* \mathcal{E})_{x'} = (f_0^* \mathcal{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ where $x = f_0(x')$ (since $\otimes$ commutes with inductive limits).

(*b*) $f^*(\mathcal{E} \underset{\mathcal{O}X}{\otimes} \mathcal{F}) = f^* \mathcal{E} \underset{\mathcal{O}X'}{\otimes} f^* \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{O}_X$-modules.

(*c*) If $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module, then $f^* \mathcal{E}$ is a coherent $\mathcal{O}_{X'}$-module.

In fact, $\mathcal{E}$ has a locally finite presentation: $\mathcal{O}_X^q \to \mathcal{O}_X^p \to \mathcal{E} \to 0$, and f^* is compatible with cokernels, $f^*(\mathcal{O}_X^r) = \mathcal{O}_{X'}^r$.

Special case: The pull-back of vector bundle. Let (E, π) be an analytic vector bundle over the analytic space X, and $f: X' \to X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X', such that $\bar{f}$ is a bundle morphism. We call this bundle E'.



Proposition 1: Let $\mathcal{E}$ (Resp. $\mathcal{E}'$) be the sheaf of analytic sections of E (Resp. E'). Then $\mathcal{E}' = f^* \mathcal{E}$.

Proof (Sketch): We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathcal{E} \to \mathcal{E}'$, which extends to a morphism $f^* \mathcal{E} \to \mathcal{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X', we can suppose that E is a trivial bundle over X with fiber $\mathbf{C}^r$, then $\mathcal{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathcal{E} = \mathcal{E}'$.

§ 3. *Introduction to flatness by examples*

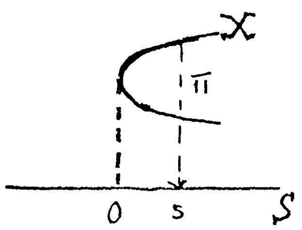
Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi: X \to S$. Let S be a simple point in S, and consider $X(s) = f^{-1}(s)$.

The main purpose of these lectures is to give a precise meaning to the expression:

" $X(s)$ depends nicely on s", and to give a criterion for the "nice" behaviour.

We begin with some examples.

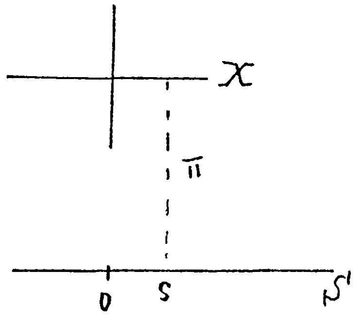
Example 1: X is the closed subspace on $\mathbf{C}^2$ defined by (y^2-x), $S = \mathbf{C}$ and π = 1st projection.



$$X(s) = \begin{cases} \text{2 simple points if } s \neq 0 \\ \text{double point if } \quad s = 0\,. \end{cases}$$

Here the behaviour of $X(s)$ is nice.

Example 2: X is the closed subspace of $\mathbf{C}^2$ defined by (xy), $S = \mathbf{C}$ and π = 1st projection.



$X(s)$ is given by $(x-s, xy)$, and

$$(x-s,\, xy) = \begin{cases} (x-s,\, y) & \text{if} \quad s \neq 0 \\ \quad (x) & \text{if} \quad s = 0\,. \end{cases}$$

The first case is a simple point, the second one the, y-axis.

A similar example is the map of a point into $\mathbf{C}$.

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreductible component of X, and after removing this component π behaves nicely.

This kind of removing is not possible in general, as the following example shows:

Example 3: X is given in $\mathbf{C}^3$ by $(xz-y)$, and π is the projection on the $(x$, y)-plane.

If $s = (x_0, y_0)$, then the fiber $X(s)$ is defined by

$$(x-x_0,\, y-y_0,\, xz-y) = \begin{cases} \left(x-x_0,\, y-y_0,\, z-\dfrac{y_0}{x_0}\right) \text{ if } x_0 \neq 0 \\ (x, y) \qquad\qquad\qquad\qquad \text{if} \quad x_0 = y_0 = 0 \\ (1) \quad \text{if} \quad x_0 = 0\ y_0 \neq 0\,. \end{cases}$$

The set of “ nice ” fibers is dense in X, so we cannot remove the z-axis and still get a closed subspace of $\mathbf{C}_3$.

§ 4. *Algebraic study of flatness*

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A-module E is *flat*, if for every exact sequence of A-modules

$$0 \to F' \to F \to F'' \to 0\,,$$

the sequence $0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is also exact. We can also say, because $\otimes$ is right exact, that E is flat, if for every injective homomorphism $F' \to F$, $E \otimes F' \to E \otimes F$ is also injective.

Examples of modules which are not flat:

(1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\,\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \to \mathbf{Z} \xrightarrow{2I} \mathbf{Z}$ $(2I : x' \to 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So $\mathbf{Z}_2$ is not a flat $\mathbf{Z}$ module.

(2) If $A = \mathbf{C}\{x\}$, $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$, $F = F' = \mathbf{C}\{x\}$, then the sequence $0 \to F \xrightarrow{xI} F'$ $(xI : p(x) \to xp(x))$ is exact. But the homomorphism $E \xrightarrow{xI} E$ is not injective.

Proposition 1: If A is an integral domain and E a flat A-module, then E is torsion-free.

Proof: Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \to AA \xrightarrow{aI}$ $(aI : x \to ax)$ is exact. Since E is flat, the sequence $0 \to E \xrightarrow{aI} E$ is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

Examples of flat modules:

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if E is free and finite type, then $E = A^n$ and $E \otimes F = F^n$. If $F' \to F$ is injective, so is $F'^n \to F^n$ too.

If E is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of E follows from (1).

(3) Let S be a multiplicative system in A. Then the ring of fractions $S^{-1} A$ is a flat A-module. In fact the ring $S^{-1} A$ can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that S has only regular elements. We can define in the set S a partial order in the following way:

$$s' \geqq s \Leftrightarrow \exists\, t \in A\,, \quad ts = s' \quad \text{(such a } t \text{ is then unique).}$$

Let $E_s = A$ for every $s \in S$, and if $s' \geqq s$ (i.e. $s' = ts$) then let $f_s^{s'}$ be the homomorphism $t.\ I_A : E_s \to E_{s'}$. The family $(E_s)_{s \varepsilon S}$ with the homomorphisms $(f_s^{s'})$ is an inductive system.

Let $E = \varinjlim E_s$ be the inductive limit of this system, and φ_s the canonical homomorphism $E_s \to E$. We shall define an isomorphism $\psi : E \to S^{-1}A$.

We first define for every s a homomorphism $\psi_s : E_s = A \to S^{-1}A$; $x \to x/s$. Now if $s' \geqq s$, then

$$(\psi_s{}' \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x)\,.$$

Therefore there exists a homomorphism $\psi : E \to S^{-1}A$, satisfying $\psi_s = \psi \circ \varphi_s$ for every $s \in S$.

Because every element of $S^{-1}A$ has the form a/s, ψ is surjective. On the other hand if $\psi(\phi_s(x)) = 0$, then $\psi_s(x) = x/s = 0$. Thus $x = 0$, and ψ is also injective.

The above proof can be extended to the general case, not assuming that the elements of S are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

Theorem 1: (Daniel, Lazard)

Any flat module is a inductive limit of free modules.

For the proof: See *C.R. Acad. Sci. Paris*, 258 (1964), pp. 6313-6316.

Some elementary properties of flat modules:

(1) If E and F are flat A-modules, then $E \otimes_A F$ is also flat. In fact, if $G' \to G$ is injective, then $F \otimes G' \to F \otimes G$ is injective, and also $E \otimes (F \otimes G') \to$ $\to E \otimes (F \otimes G)$ is injective. The result follows from the assosiativity of the tensor product.

(2) Let $\phi : A \to B$ be a ring homomorphism, and E a flat A-module. The module $B \otimes_A E$ is a flat B-module.

If F is a B-module, then $F \otimes_B (B \otimes_A E) = (F \otimes_B B) \otimes_A E = F \otimes_A E$ further if F' and F are B-modules, and $F' \to F$ an injective homomorphism of B-modules, we can consider this homomorphism as an injective homomorphism of A-modules. Because E is A-flat,

$$F' \otimes_A E \to F \otimes_A E \text{ is injective.}$$

(3) Let $\phi : A \to B$ be a ring homomorphism, such that B is a flat A-module. If F is a flat B-module, then F is a flat A-module. In fact: if $E' \to E$ is injective, then $E' \otimes_A B \to E \otimes_B B$ is injective, and also $(E \otimes_A B) \otimes_B F' \to (E \otimes_A B) \otimes_B F$ is injective. But $(E' \otimes_A B)_B \otimes F' = E' \otimes_A F$; $(E \otimes_A B) \otimes_B F = E \otimes_A F$.

If an A-module E is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor.

Definition 2: A free resolution of E is an exact sequence: $\ldots \to L_n \to L_{n-1} \to$ $\to \ldots \to L_1 \to L_0 \to E \to 0$, where all L_i are free A-modules.

The complex of the resolution is the sequence

$$(L.) \ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to 0 .$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensorproducts $L_i \otimes F$, we get

$$(L. \otimes F) \ldots \to L_n \otimes F \to L_{n-1} \otimes F \to \ldots \to L_1 \otimes F \to L_0 \otimes F \to 0 .$$

Definition 3:

$$\mathrm{Tor}_n^A (E, \mathrm{F}) = H_n (L. \otimes F) = \frac{\mathrm{Ker}(L_n \otimes F \to L_{n-1} \otimes F)}{\mathrm{Im}\,(L_{n+1} \otimes F \to L_n \otimes F)}$$

if $n \geqq 1$, and $\mathrm{Tor}_0^A (E, F) = \mathrm{Coker}\,(L_1 \otimes F \to L_0 \otimes F) = E \otimes F.$

Basic properties of Tor:

(1) $\mathrm{Tor}_n (E, F)$ is independent of the choice of the resolution (up to a canonical isomorphism).

(2) If we take a free resolution of F, we get $\mathrm{Tor}_n(F, E) = \mathrm{Tor}_n(E, F)$ (Symmetry of the Tor). We can also define $\mathrm{Tor}_n(E, F)$ by taking two free resolutions, one of E and one of F.

(3) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence, then we get a long exact sequence:

$$\begin{aligned} &\mathrm{Tor}_n(E', F) \to \mathrm{Tor}_n(E, F) \to \mathrm{Tor}_n(E'', F) \to \\ \to\ &\mathrm{Tor}_{n-1}(E', F) \to \mathrm{Tor}_{n-1}(E, F) \to \mathrm{Tor}_{n-1}(E'', F) \to \\ \to\ &\text{— — — — — — — — — — — — — — — — —} \to \\ \to\ &\mathrm{Tor}_1(E', F) \to \mathrm{Tor}_1(E, F) \to \mathrm{Tor}_1(E'', F) \to \\ \to\ &E' \otimes F \to E \otimes F \to E'' \otimes F \to 0\,. \end{aligned}$$

(4) Tor is compatible with inductive limit, i.e. if $E = \varinjlim (E_i)$, then

$\mathrm{Tor}_n(\varinjlim E_i, F) = \varinjlim (\mathrm{Tor}_n(E_i, F))$.

(5) We can define $\mathrm{Tor}_n(E, F)$ by taking a flat resolution of E.

Proposition 3: Let E be an A-module. Then the following conditions are equivalent:

(a) E is flat.

(b) For all A-modules F, and for all $n \geqq 1$, $\mathrm{Tor}_n(E, F) = 0$.

(c) For all A-modules F, $\mathrm{Tor}_1(E, F) = 0$.

Proof: (a) $\Rightarrow$ (b). If $\ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to F \to 0$ is a free resolution of F, then the sequence

$$\ldots \to E \otimes L_n \to E \otimes L_{n-1} \to \ldots \to E \otimes L_1 \to E \otimes L_0 \to E \otimes F \to 0$$

is exact, thus $\mathrm{Tor}_n(E, F) = 0$ for all $n \geqq 1$.

(b) $\Rightarrow$ (c) clear. (c) $\Rightarrow$ (a): If the sequence $0 \to F' \to F \to F'' \to 0$ is exact, so is also (by (3) above) $\mathrm{Tor}_1(E, F'') \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$. Now $\mathrm{Tor}_1(E, F'') = 0$, thus E is flat.

Proposition 4: If I and J are two ideals in A, then $\mathrm{Tor}_1^A(A/I, A/J) = I \cap J / I . J$.

Proof: From the exact sequence $0 \to I \to A \to A/I \to 0$, we get the exact sequence:

$$\mathrm{Tor}_1(A, A/J) \to \mathrm{Tor}_1(A/I, A/J) \to I \otimes A/J \to A \otimes A/J \to A/I \otimes A/J \to 0\,.$$

But now $\mathrm{Tor}_1(A, A/J) = 0$ (A beeing A-free), and $I \otimes A/J = I/I . J$; $A \otimes A/J = A/J$. Therefore the sequence $0 \to \mathrm{Tor}_1(A/I, A/J) \to I/I . J \to A/J$ is exact, and $\mathrm{Tor}_1(A/I, A/J) = \mathrm{Ker}(I/I . J \to A/J) = I \cap J/I . J$.

Example : Let U be an open set in $\mathbf{C}^n$, and $x \in U$. Further let $X, Y \subset U$ be two hypersurfaces, defined by $I = (f)$ and $J = (g)$. Supposing that f and g do not have common factors: $I_x \cap J_x = I_x J_x$, and

$$\operatorname{Tor}_1 (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = \operatorname{Tor}_1 (\mathcal{O}_{U,x}/I_x , \quad \mathcal{O}_{U,x}/J_x) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0 .$$

Heuristic remark : The formula $\operatorname{Tor}_1 (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ expresses the fact that X and Y are " in general position ". If for example X and Y are two linears subspaces in $\mathbf{C}^n$ of dimensions p and q, we have $\operatorname{Tor}_1 (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ if $\dim (X \cap Y) = p+q-n$, and $\operatorname{Tor} (\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$ otherwise.

Next we shall prove an elementary flatness criterion.

Proposition 5 : Let E be an A-module. The following conditions are equivalent:

(a) E is flat.

(b) For all finitely generated ideals I of A, $\operatorname{Tor}_1 (E, A/I) = 0$.

(c) For all monogenous A-modules F, $\operatorname{Tor}_1 (E, F) = 0$.

Proof: (a) $\Rightarrow$ (b), by prop. 3.

(b) $\Rightarrow$ (c): Because Tor is compatible with inductive limit, we can suppose, that $\operatorname{Tor}_1 (E, A/I) = 0$ for an arbitrary ideal I of A. But every monogenous A-module F can be represented by A/I.

(c) $\Rightarrow$ (a). By prop. 3 it is sufficient to prove that $\operatorname{Tor}_1 (E, F) = 0$ for any A-module F.

First consider the case, where F is finitely generated. We use induction, supposing that $\operatorname{Tor}_1 (E, F) = 0$, when F has n generators. Let F have $(n+1)$ generators $x_1, \ldots, x_n, x_{n+1}$. If F' is the submodule generated by $\{x_1, \ldots, x_n\}$, then $F' \subset F$ and $F'' = F/F'$ is monogenous. The exact sequence $0 \to F' \to F \to F'' \to 0$ gives the exact sequence $\operatorname{Tor}_1 (E, F') \to \operatorname{Tor}_1 (E, F) \to \operatorname{Tor}_1 (E, F'')$. Now $\operatorname{Tor}_1 (E, F') = \operatorname{Tor}_1 (E, F'') = 0$, thus $\operatorname{Tor}_1 (E, F) = 0$. In the general case, F can be considered as an inductive limit of finitely generated modules, and because Tor is compatible with inductive limits, $\operatorname{Tor}_1 (E, F) = 0$.

Proposition 6 : Let A be an integral domain, and E an A-module. Then E is torsionfree if and only if $\operatorname{Tor}_1 (E, A/(a)) = 0$, for any element $a \in A$.

Proof: If E is A-module, $a \in A$, then the exact sequence $0 \to A \xrightarrow{aI} A \to A/(a) \to 0$ gives the exact sequence $0 \to \operatorname{Tor}_1 (E, A/(a)) \to E \xrightarrow{aI} E$. In other words $\operatorname{Tor}_1 (E, A/(a)) = \{x \in E \mid ax = 0\}$, from which the result follows.

Corollary: Let A be a principal ideal domain. E is flat if and only if E is torsionfree.

Proof: We have already proved that, if E is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

Theorem 2: Let A be a noetherian local ring with maximal ideal m; $k = A/m$, and E a finitely generated A-module. The following conditions are equivalent:

(*a*) E is free.

(*b*) E is flat.

(*c*) $\operatorname{Tor}_1^A(E, k) = 0$.

Proof: We have already proved $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$: We recall first Nakayma's lemma. If A is a local ring with maximal ideal m; $k = A/m$, and E is a finitely generated A-module, such that $k \otimes_A E = E/mE = 0$, then $E = 0$.

The module $\bar{E} = k \otimes_A E = E/mE$ is a finitely generated vector space over k. Let $\{\bar{x}_1. \ldots, \bar{x}_r\}$ be a base of $\bar{E}$ (over k), and $\{x_1, \ldots, x_r\}$ E representatives of $\bar{x}_i : s$. Consider the homomorphism $\phi : A^r \to E$, $\phi(a_1, \ldots, a_r) = \sum a_i x_i$. Denoting by R and Q the kernel and the cokernel of ϕ, we get an exact sequence:

$$(*) \qquad 0 \to R \to A^r \to E \to Q \to 0$$

and R, Q are finitely generated A-modules. From (*) we get the exact sequence

$$A^r \otimes_A k \to E \otimes_A k \to Q \otimes_A k \to 0 .$$

But $\bar{E} = E \otimes_A k \simeq k^r = A^r \otimes_A k$, so $Q \otimes_A k = 0$, and by Nakayama's lemma $Q = 0$.

Therefore ge have an exact sequence

$$0 \to R \to A^r \to E \to 0 .$$

From this we get: $\operatorname{Tor}_1(E, k) \to k \otimes_A R \to k^r \to \bar{E} \to 0$ (exact). Now: $\bar{E} \simeq k^r$, $\operatorname{Tor}_1(E, k) = 0$ (by assumption). Therefore $k \otimes_A R = 0$, and once more by Nakayama's lemma $R = 0$, thus $E \simeq A^r$, i.e. E is free.

Proposition 7: Let $\phi : A \to B$ be a ring homomorphism, and let B be A-flat. If I is an ideal of A, we write $\bar{A} = A/I$, $\bar{B} = B/IB = \bar{A} \underset{A}{\otimes} B$. Let F be a B-module, then: $\mathrm{Tor}_i^A(\bar{A}, F) = \mathrm{Tor}_i^B(\bar{B}, F)$ $(i \geqq 0)$.

Proof: We choose first a B-free resolution of F

$$\to L_{n+1} \to L_n \to \ldots \to L_1 \to L_0 \to F \to 0 .$$

If $L.$ is the respective complex of resolution, then

$$\bar{B} \underset{B}{\otimes} L. = B/IB \underset{B}{\otimes} L. = \bar{A} \underset{A}{\otimes} (B \underset{B}{\otimes} L.) = \bar{A} \underset{A}{\otimes} L.$$

Because every L_i is B-free, and B is A-flat, every L_i is A-flat (Property 3 after Th. 1). Thus $L.$ is a flat A-resolution, and

$$\mathrm{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \underset{A}{\otimes} L.) = H_i(\bar{B} \underset{B}{\otimes} L.) = \mathrm{Tor}_i^B(\bar{B}, F) .$$

We shall next state the second flatness criterion for noetherian local rings.

Theorem 3: Let A and B be two noetherian local rings, with maximal ideals $\underline{m}$, $\underline{n}$; $k = A/\underline{m}$. If $\phi : A \to B$ is a local homomorphism (i.e. $\phi(\underline{m}) \subset \underline{n}$), and F finitely generated B module then

$$F \text{ is A-flat} \Leftrightarrow \mathrm{Tor}_1^A(k, F) = 0 .$$

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, $(i) \Leftrightarrow (iii)$, p. 98.

The conditions in Bourbaki's theorem are here fullfilled:

1° A finitely generated module F over a noetherian local ring B is idealwise separated for $\underline{n}$. (*Ibid.*, § 5. 1. Ex. 1, p. 97.)

2° If $\phi : A \to B$ is a local homomorphism, F is also idealwise separated for $\underline{m}$. (*Ibid.*, § 5, prop. 2, p. 101.)

3° Also the flatness condition is fulfilled, because k is a field.

Remark: The main interest of the theorem lies in the fact, that it is true without any assumption of finitness on B.

Corollary: If the assumptions are the same as in the theorem 3, and if moreover B is A-flat, then

$$F \text{ is } A\text{-flat} \Leftrightarrow \mathrm{Tor}_1^B(\bar{B}, F) = o ,$$

where $\bar{B} = B/mB$.

Proof: $\mathrm{Tor}_1^A(k, F) = \mathrm{Tor}_1^B(\bar{B}, F)$, by prop. 7.

§ 5. *Geometric applications of the flatness criterions*

A) *Flatness for finite morphisms*

Proposition 1: Let $\pi\colon X \to S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_*(\mathcal{O}_X)$ is a coherent analytic sheaf over S. The following conditions are equivalent:

(*a*) π is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, $s = \pi(x)$).

(*b*) For every s, $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$-module.

(*c*) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

Proof: Because π is finite $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \varepsilon \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is $(b) \Rightarrow (c)$.

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and $\mathcal{E}$ a coherent $\mathcal{O}_s$-module. Let $E(s)$ be the finite dimensional vector space (over **C**) $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s$. $\mathcal{E}$ is a locally free $\mathcal{O}_{S,s}$-module if an only if $\dim_{\mathbf{C}} E(s)$ is locally constant.

Proof: If $\mathcal{E}$ is locally free, then $\dim_{\mathbf{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbf{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \to \mathcal{E}_U \to 0$ is exact. d is determined by a $p \times q$ matrix of analytic functions on U, so it gives a morphism $\mathbf{C}_U^p \xrightarrow{d} \mathbf{C}_U^q$ of trivial vector bundles over U.

From the exact sequence $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \to \mathcal{E}_s \to 0$, we get (by making tensor-products with $\mathbf{C}_s$) the exact sequence:

$$\mathbf{C}_s^p \xrightarrow{d(s)} \mathbf{C}_s^q \to E(s) \to 0\,,$$

which shows that d has constant rank in U. Thus Ker d and Im d are vector bundles, and we can write

$$\mathbf{C}_U^p = F_1 \oplus G_1\,, \quad \mathbf{C}_U^q = F_0 \oplus G_0\,,$$

$$d : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_0\,. \end{cases}$$

Now $\mathscr{E} \simeq$ the sheaf of analytic sections of G_0, therefore $\mathscr{E}$ is locally free.

Definition 1: Let $\pi : X \to S$ be a finite morphism of analytic spaces, and $s \in S$. For each $x \in X(s) = \pi^{-1}(s)$, $\mathcal{O}_{X(s),x} = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is finite dimensional vectorspace over $\mathbf{C}$. Denote its dimension by $\nu(x)$. Then the degree $\nu(s)$ of s is defined by $\nu(s) = \sum_{x \varepsilon X(s)} \nu(x)$.

Theorem 1: Let $\pi : X \to S$ be a finite morphism of analytic space and let S be a reduced space. Then X is flat over S if and only if $\nu(s)$ is locally constant function of s.

Proof:
$$\nu(s) = \sum_{x \varepsilon X(s)} \dim_{\mathbf{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbf{C}} \Big(\bigoplus_{x \varepsilon X(s)} \mathcal{O}_{X(s),x} \Big)$$
$$= \dim_{\mathbf{C}} \Big(\bigoplus_{x \varepsilon X(s)} (\mathbf{C} \otimes_{\mathcal{O}_{S \cdot s}} \mathcal{O}_{X,x}) \Big)$$
$$= \dim_{\mathbf{C}} \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \pi_* (\mathcal{O}_X)_s = \dim_{\mathbf{C}} E(s) \,.$$

The theorem follows from propositions 1 and 2.

Examples of flat morphisms

Example 1: If $\pi : X \to S$ is a local isomorphism near x, then π is flat at x.

Example 2: Consider § 2, Ex. 1. Here $\nu(x) = 1$.

Examples of non-flat morphisms

Examples 1: If $X \subset S$ is a closed subspace, not open, $\nu(s)$ is not locally constant.

Example 2: Let X be a subspace of $\mathbf{C}^4$ defined by the ideal intersection of (x_3, x_4) and $(x_1 - x_1, x_4 - x_2)$ (which is equal to the product ideal) and let π be the projection onto the (x_1, x_2) – plane $\mathbf{C}^2$. Then X is a union of two 2-planes in $\mathbf{C}^4$, whose intersection is (0). When $s \neq 0$. $X(s)$ consists of two simple points, so $\nu(s) = 2$. $X(0)$ is given by the ideal $(x_1, x_2, x_3^2, x_3 x_4, x_4^2)$, thus $\nu(0) = 3$.

Example 3: Let $S = \{(u, v, w) \in \mathbf{C}^3 \mid v^2 = uw\}$ and $\pi : \mathbf{C}^2 \to S$ be the map $(x, y) \to (x^2, xy, y^2)$. This map identifies S with the quotient of $\mathbf{C}^2$ by the equivalence relation idenfying (x, y) with $(-x, -y)$. However, π is not flat, since for $s \in S$, $\nu(s) = 2$ if $s \neq 0$ and $\nu(s) = 3$ if $s = 0$.

B) *Projection of a product of analytic spaces*

Theorem 2: Let S and X be analytic spaces. If $\pi : S \times X \to S$ is the projection morphism, then π is flat, i.e. $\mathcal{O}_{S\times X,(s,x)}$ is a flat $\mathcal{O}_{S,s}$ module for every $(s, x) \in S \times X$.

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when S is a manifold, and finally in the general case.

(a) *Koszul complex*

Let A be a ring, M an A-module and $h_1, \dots, h_n$ homomorphisms $M \to M$, which commute with each other, i.e. $h_i h_j = h_j h_i$ for every i, j.

If $1 \leqq k \leqq n$, set $Q_k = M/h_1(M) + \dots + h_k(M)$, and $Q_0 = M$, thus, in particular, $Q_n = Q = M/\sum_{i=j}^{n} h_i(M)$,. Every h_k induces a map $\tilde{h}_k\, Q_{k-1} \to Q_{k-1}$.

Definition 2: The sequence $(h_1, \dots, h_n)$ is called regular if each of the mappings $\tilde{h}_k$ $(1 \leqq k \leqq n)$ is injective.

The Koszul complex of the module M and of the mappings h_k $(1 \leqq k \leqq n)$ $K. = K.[M; h_1, \dots, h_n]$ is defined in the following way:

$$K_i = \wedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \qquad 0 \leqslant i \leqslant n .$$

We define the homorphisms $d_i : K_i \to K_{i-1}$ $(i > 0)$ by $\lambda \otimes x \to \sum_i (e_i \wedge \lambda) \otimes \otimes h_i(x)$, where (e_i) is the natural base of A^n. We also define $\varepsilon : K_0 \to Q$ as the natural map : $K_0 = M \to M/\sum_{i=1}^{n} h_i(M) = Q$. Using the fact that $h_1, \dots, h_n$ commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also $\varepsilon d_1 = 0$. Thus $K.$ is really a complex.

Theorem 3 (Poincaré-Koszul).

If $(h_1, \dots, h_n)$ is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & if \quad i = 0 \\ 0 & if \quad i > 0 \end{cases} .$$

If $h_i \in A$, it defines the map: $A \xrightarrow{h_i I} A$, which we denote also by h_i. We say that $(h_1, ..., h_n)$ is a regular sequence of elements if $(h_1 I, ..., h_n I)$ is a regular sequence.

Corollary. If $(h_1, ..., h_n)$ is a regular sequence of elements, then the Koszul complex $K. = K.\,[A; h_1, ..., h_n] = \{ \wedge^{n-1} A^n \simeq A^{\binom{n}{i}} \}$ is a free resolution of $Q = A/(h_i)$ ((h_i) is the ideal generated by $h_1, ..., h_n$)).

Example: If $A = \mathbf{C}\{x_1, ..., x_n\}$; $h_i = x_i$, then $Q_k = A/(x_1, ..., x_k) = \mathbf{C}\{x_{k+1}, ..., x_n\}$ and $Q = Q_n = \mathbf{C}$. The complex $K. = K.\,[A; x_1, ..., x_n]$ is a free resolution of $\mathbf{C}$.

(b) *Proof of theorem 2, when S is a complex manifold*

In this case we can take $\mathcal{O}_{S,s} = \mathbf{C}\{t_1, ..., t_m\} = A$ and if $\mathcal{O}_{X,x} = \mathbf{C}\{x_1, ..., x_n\}/(f_1, ..., f_p)$, then

$$\mathcal{O}_{S\times X,(s,x)} = \mathbf{C}\{t_1, ..., t_m, x_1, ..., x_n\}/(f_1, ..., f_p) = B\,.$$

B is an A-module in a natural way.

By the corollary of the Poincaré-Koszul theorem $K. = K.\,[A; t_1, ..., t_m]$ in a free resolution of $\mathbf{C}$. We want to compute the modules $\mathrm{Tor}_i^A(\mathbf{C}, B) = H_i(K.\otimes B)$ $(i>0)$.

It's easily seen, that we can consider the complex $K.\otimes B$ as a Koszul complex $K'. = K.\,[B; t_1, ..., t_m]$ (where $t_i : B \xrightarrow{t_i I} B$). But now the sequence $(t_1, ..., t_m)$ is regular, thus by the Poincaré-Koszul theorem $H_i\,[K'.] = 0$ if $i>0$.

In particular: $\mathrm{Tor}_1^A(\mathbf{C}, B) = H_1\,[K.\otimes B] = H_1\,[K'.] = 0$. By the second flatness criterion B is A-flat.

(c) *The general case*

The question being local, we can suppose that $S \subset W \subset \mathbf{C}^n$, where W is open, and S an analytic subspace of W. Let S be defined by $g_1, ..., g_r$. Then $S\times X \subset W\times X$ and $\mathcal{O}_S = \mathcal{O}_W/(g_1, ..., g_r)$. On the other hand $\mathcal{O}_{S\times X} = \mathcal{O}_{W\times X}/(g_1, ..., g_r) = \mathcal{O}_S \otimes_{\mathcal{O}_W} \mathcal{O}_{W\times X}$. The last equality follows from the fact, that if $\pi : X \to S$ is a morphism, and $S' \subset S$ a subspace, $X' = \pi^{-1}(S')$, then $\mathcal{O}_{X'} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X$.

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & S & \end{array} \qquad \mathcal{O}_{X \underset{S}{\times} Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_y \, .$$

We have proved that $\mathcal{O}_{W \times X}$ is $\mathcal{O}_W$-flat, so by scalar extension $\mathcal{O}_{S \times X}$ is $\mathcal{O}_S$ flat.

Corollary : If X and S are two manifolds and $\pi : X \to S$ is a submersion, then π is flat.

III. Privileged polycylinders

§ 1. *Banach vector bundles over an analytic space*

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E. If $U \subset \mathbf{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf $\mathcal{O}_U/J$, where $U \subset \mathbf{C}^n$ is open and J is a coherent sheaf of ideals of $\mathcal{O}_U$, then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathcal{H}(V, E)/J_V \, . \, \mathcal{H}(V, E)$ ($V \subset U$, V-open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X, i.e. $\mathcal{H}_X(E)|X' = \mathcal{H}_{X'}(E)$, for every local model X'.

Definition 1 : The set of *analytic morphisms* from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2 : An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.