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Remark : The inverse image is a particular case of the analytic pull-back.

In fact, if Y is a closed analytic subspace of X and $f : X' \rightarrow X$ is a morphism:

$$f^* \mathcal{O}_Y = f_0^* (\mathcal{O}_X / J_Y) \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \\ \simeq \mathcal{O}_{X'} / f^1(J_Y). \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that $A/I \otimes_A E \simeq E/IE$).

Elementary properties of the analytic pull-back :

- (a) $(f^* \mathcal{E})_{x'} = (f_0^* \mathcal{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ where $x = f_0(x')$ (since \otimes commutes with inductive limits).
- (b) $f^* (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f^* \mathcal{E} \otimes_{\mathcal{O}_{X'}} f^* \mathcal{F}$, where \mathcal{E} and \mathcal{F} are \mathcal{O}_X -modules.
- (c) If \mathcal{E} is a coherent \mathcal{O}_X -module, then $f^* \mathcal{E}$ is a coherent $\mathcal{O}_{X'}$ -module.

In fact, \mathcal{E} has a locally finite presentation:

$$\mathcal{O}_X^q \rightarrow \mathcal{O}_X^p \rightarrow \mathcal{E} \rightarrow 0, \text{ and } f^* \text{ is compatible with cokernels, } f^* (\mathcal{O}_X^r) = \mathcal{O}_{X'}^r.$$

Special case : The pull-back of vector bundle. Let (E, π) be an analytic

$$\begin{array}{ccc} E \times X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

vector bundle over the analytic space X , and $f : X' \rightarrow X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X' , such that \bar{f} is a bundle morphism. We call this bundle E' .

Proposition 1 : Let \mathcal{E} (Resp. \mathcal{E}') be the sheaf of analytic sections of E (Resp. E'). Then $\mathcal{E}' = f^* \mathcal{E}$.

Proof (Sketch) : We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathcal{E} \rightarrow \mathcal{E}'$, which extends to a morphism $f^* \mathcal{E} \rightarrow \mathcal{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X' , we can suppose that E is a trivial bundle over X with fiber \mathbf{C}^r , then $\mathcal{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathcal{E} = \mathcal{E}'$.

§ 3. Introduction to flatness by examples

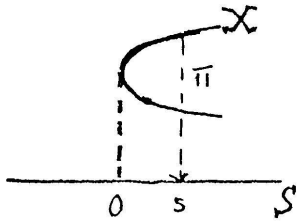
Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi : X \rightarrow S$. Let S be a simple point in S , and consider $X(s) = f^{-1}(s)$.

The main purpose of these lectures is to give a precise meaning to the expression:

“ $X(s)$ depends nicely on s ”, and to give a criterion for the “ nice ” behaviour.

We begin with some examples.

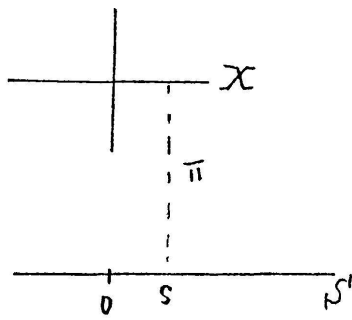
Example 1: X is the closed subspace on \mathbf{C}^2 defined by $(y^2 - x)$, $S = \mathbf{C}$ and $\pi = 1st$ projection.



$$X(s) = \begin{cases} 2 \text{ simple points if } s \neq 0 \\ \text{double point if } s = 0. \end{cases}$$

Here the behaviour of $X(s)$ is nice.

Example 2: X is the closed subspace of \mathbf{C}^2 defined by (xy) , $S = \mathbf{C}$ and $\pi = 1st$ projection.



$X(s)$ is given by $(x-s, xy)$, and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if } s \neq 0 \\ (x) & \text{if } s = 0. \end{cases}$$

The first case is a simple point, the second one the y -axis.

A similar example is the map of a point into \mathbf{C} .

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreducible component of X , and after removing this component π behaves nicely.

This kind of removing is not possible in general, as the following example shows:

Example 3: X is given in \mathbf{C}^3 by $(xz - y)$, and π is the projection on the (x, y) -plane.

If $s = (x_0, y_0)$, then the fiber $X(s)$ is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\frac{y_0}{x_0}\right) & \text{if } x_0 \neq 0 \\ (x, y) & \text{if } x_0 = y_0 = 0 \\ (1) & \text{if } x_0 = 0, y_0 \neq 0. \end{cases}$$

The set of “ nice ” fibers is dense in X , so we cannot remove the z -axis and still get a closed subspace of \mathbf{C}_3 .

§ 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A -module E is *flat*, if for every exact sequence of A -modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0,$$

the sequence $0 \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is also exact. We can also say, because \otimes is right exact, that E is flat, if for every injective homomorphism $F' \rightarrow F$, $E \otimes F' \rightarrow E \otimes F$ is also injective.

Examples of modules which are not flat :

- (1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \rightarrow \mathbf{Z} \xrightarrow{2I} \mathbf{Z} (2I : x' \rightarrow 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So \mathbf{Z}_2 is not a flat \mathbf{Z} module.
- (2) If $A = \mathbf{C}\{x\}$, $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$, $F = F' = \mathbf{C}\{x\}$, then the sequence $0 \rightarrow F \xrightarrow{xI} F' (xI : p(x) \rightarrow xp(x))$ is exact. But the homomorphism $E \xrightarrow{xI} E$ is not injective.

Proposition 1: If A is an integral domain and E a flat A -module, then E is torsion-free.

Proof: Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$ is exact. Since E is flat, the sequence $0 \rightarrow E \xrightarrow{aI} E$ is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

Examples of flat modules :

- (1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.