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Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that $\mathcal{O}_{W \times X}$ is \mathcal{O}_W -flat, so by scalar extension $\mathcal{O}_{S \times X}$ is \mathcal{O}_S flat.

Corollary : If X and S are two manifolds and $\pi : X \rightarrow S$ is a submersion, then π is flat.

III. PRIVILEGED POLYCYLINDERS

§ 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X .

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E . If $U \subset \mathbb{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \rightarrow E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X , i.e. X' is the support of the quotient sheaf \mathcal{O}_U/J , where $U \subset \mathbb{C}^n$ is open and J is a coherent sheaf of ideals of \mathcal{O}_U , then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$ ($V \subset U$, V -open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U .

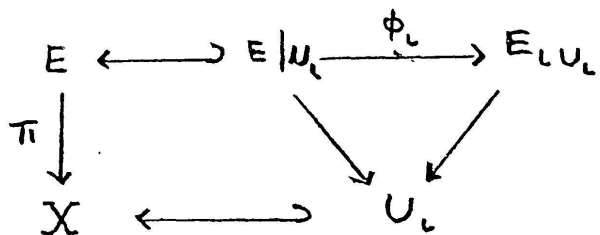
The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X , i.e. $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$, for every local model X' .

Definition 1 : The set of analytic morphisms from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F .

Definition 2 : An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi : E \rightarrow X$ a continuous projection.



Suppose that X has an open covering $(U_l)_{l \in I}$, and that for every $l \in I$ there is given a trivial Banach space bundle $E|_{U_l}$ and a homeomorphism ϕ_l , such that the following diagram is commutative:

We suppose further that for each pair $l, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{l\kappa} : E|_{U_l \cap U_\kappa} \rightarrow E|_{U_l \cap U_\kappa}$, with the underlying mapping $\phi_l \circ \phi_\kappa^{-1}$, such that:

$$\gamma_{l\lambda} = \gamma_{l\kappa} \gamma_{\kappa\lambda}; \quad \gamma_{ll} = I, \quad \text{for all } l, \kappa, \lambda \in I.$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{l\kappa}$ are determined by their underlying map and the condition $\gamma_{l\lambda} = \gamma_{l\kappa} \gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\phi : E \rightarrow F$ be a morphism of two Banach vector



bundles E and F , and $x \in X$.

If $\phi_x \in \mathcal{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x , such that $\phi|_U : E|_U \rightarrow F|_U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|_V = E_0|_V, F|_V = F_0|_V$ at $x \in V \subset X$ (V -open).

The set $\text{Isom}(E_0, F_0)$ of isomorphic mappings is an open subset of $\mathcal{L}(E_0, F_0)$ and the mapping $g \rightarrow g^{-1}$ is an analytic isomorphism:

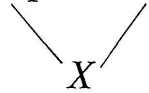
$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0).$$

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \rightarrow \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|_U)^{-1} : F|_U \rightarrow E|_U$.

Definition 3 : Let E and F be two Banach spaces and f a continuous linear mapping from E into F . f is a *split mono-(epi) morphism*, if there exists a mapping $g \in \mathcal{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definirion 4 : Let E_1 and E_2 be two Banach vector bundles over an analytic space X , and f a vector bundle morphism from E_1 into E_2 . f is a *split mono (epi) morphism*, if there exists a vector bundle morphism $g : E_2 \rightarrow E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f : E_1 \rightarrow E_2$ is a split monomorphism if and only if E_2 can



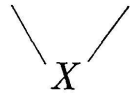
be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \rightarrow G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that } f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq E_2 \end{cases}.$$

Proposition 2 : Let $E \xrightarrow{\phi} F$ be a bundle morphism and $x \in X$.



If $\phi_x : E(x) \rightarrow F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi|_U : E|_U \rightarrow F|_U$ is a split vector bundle epi (mono) morphism.

Proof : Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|_V = E_{0V}, F|_V = F_{0V}$ at x , so that there exists a mapping $\sigma \in \mathcal{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi : F_{0V} \rightarrow E_{0V}$ by $x \rightarrow \sigma \in \mathcal{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi : F_{0V} \rightarrow E_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in x . By proposition 1 we have an isomorphic restriction $\gamma|_U, \phi|_U \circ (\psi|_U \circ (\gamma|_U)^{-1}) = I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5 : Let B_1, B_2, B_3 be Banach spaces, and $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms a *complex*, if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct

sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \rightarrow 0 \\ D_1 \simeq C_2 \end{cases} \quad k: \begin{cases} C_2 \rightarrow 0 \\ D_2 \simeq C_3 \end{cases} .$$

Definition 6: A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \\ & & X & & \end{array} \quad \text{is a complex if } g \circ f = 0.$$

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_2 \end{cases} \quad g: \begin{cases} F_2 \rightarrow 0 \\ G_2 \simeq F_3 \end{cases} .$$

Theorem 1: Let $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \end{array}$ be a complex of Banach vector

bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \xrightarrow{f_{x_0}} E_2(x_0) \xrightarrow{g_{x_0}} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0 , such that $E_1|_U \xrightarrow{f|_U} E_2|_U \xrightarrow{g|_U} E_3|_U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x , such that we have a complex $E_{1V} \xrightarrow{f|_V} E_{2V} \xrightarrow{g|_V} E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0}: \begin{cases} F_1(x_0) \rightarrow 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \quad g_{x_0}: \begin{cases} F_2(x_0) \rightarrow 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2, $f|_V : G_{1V} \rightarrow E_{2V}$, $g|_V : G_{2V} \rightarrow E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W} , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W} .$$

By our construction

$$g|_W : \begin{cases} F_2 & \rightarrow 0 \\ G_2 W & \simeq F_3 \end{cases} .$$

If $p: E_{2W} \rightarrow F_2$ is the projection with kernel G_{2W} , the map, $p \circ f: E_{1W} \rightarrow F_2$ is a split epimorphism in x_0 . Again by prop. 2 we have over an open neighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \text{Ker } p \circ f$)

$$(p \circ f)|_U : \begin{cases} F_1 & \rightarrow 0 \\ G_{1U} & \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image $f|_U(F_1)$ is contained in G_{2U} . But $g|_U \circ f|_U = 0$ and $g|_{G_{2U}}$ is a monomorphism hence $f|_U: F_1 \rightarrow 0$. We get finally (restricting all our morphisms to U)

$$f|_U : \begin{cases} F_{1U} & \rightarrow 0 \\ G_{1U} & \simeq F_{2U} \end{cases} \qquad g|_U : \begin{cases} F_{2U} & \rightarrow 0 \\ G_{2U} & \xrightarrow{\sim} F_{3U} \end{cases} .$$

§ 2. Privileged polycylinders

Definition 1: A polycylinder in \mathbf{C}^n is a compact set K of the form $K = K_1 \times \dots \times K_n$ where each K_i is a compact, convex subset of \mathbf{C} , with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of \mathbf{C}^n . Let \mathcal{F} be a coherent analytic sheaf on U .

(A) There exists an open neighbourhood of K over which \mathcal{F} admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

(B) $H^q(K, \mathcal{F}) = 0$ for $q > 0$.

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf \mathcal{F} , the sequence

$$0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0$$

is an $\mathcal{O}_U(K)$ - free resolution of $\mathcal{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0,$$

then the sequence

$$0 \rightarrow \mathcal{F}'(K) \rightarrow \mathcal{F}(K) \rightarrow \mathcal{F}''(K) \rightarrow 0 \quad \text{is exact.}$$

Let \mathcal{F} be a coherent analytic sheaf on U , and let $K \subset U$ be a polycylinder. If V is an open neighbourhood of K , then $\mathcal{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathcal{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathcal{F}(K)$ and by choosing K in a "privileged" way.

Let $B(K) = \{f: K \rightarrow \mathbb{C} \mid f \text{ continuous on } K \text{ and analytic on } \overset{\circ}{K}\}$, then $B(K)$ is Banach algebra and $B(K) \subset C(K)$. The sections of \mathcal{O}_U over K are elements of $B(K)$, and $B(K)$ is in fact the uniform closure of $\mathcal{O}_U(K)$ in $C(K)$.

If $\mathcal{L} = \mathcal{O}_U^r$, we define $B(K, \mathcal{L}) = B(K)^r$. Then $B(K; \mathcal{L})$ is a free $B(K)$ -module, and since $\mathcal{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathcal{L}) = B(K) \otimes_{\mathcal{O}_U(K)} \mathcal{L}(K)$.

We now assume that \mathcal{F} is a coherent sheaf on U , where $U \subset \mathbb{C}^n$ is open. Consider a free resolution

$$(R) \quad 0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 \quad \text{of } \mathcal{F}.$$

From (R) we get an $\mathcal{O}_U(K)$ -free resolution of $\mathcal{F}(K)$

$$(R') \quad 0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_1(K) \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_U(K)}$ we get the complex

$$B(K; \mathcal{L}.) : 0 \rightarrow B(K; \mathcal{L}_n) \rightarrow \dots \rightarrow B(K; \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0).$$

Definition 2: The polycylinder K is called \mathcal{F} -privileged if the complex $B(K; \mathcal{L}.)$ is split-exact in every degree > 0 .

Remark: The property of being \mathcal{F} -privileged is independent of the resolution (R).

The exactness of $B(K; \mathcal{L}.)$ can be expressed by $\text{Tor}_i^{\mathcal{O}_U(K)}(B(K), \mathcal{F}(K)) = 0$, for every $i > 0$, and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of (R) , and this is omitted.

Since $B(K; \mathcal{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K is \mathcal{F} -privileged. In this case we define $B(K; \mathcal{F}) = \text{Coker } (B(K, \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{O}_U} \mathcal{F}(K)$ and we get a $B(K)$ -module, which is a Banach-space.

Warning: In the definition of split-exactness, the subspaces are splitting vector spaces, but they are not splitting $B(K)$ -modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of \mathbb{C}^n , and let \mathcal{F} be a coherent analytic sheaf on U . For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U , which are \mathcal{F} -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in \mathbb{C}^2) Let $U \subset \mathbb{C}^2$ be an open connected neighbourhood of the origin, and let $h: U \rightarrow \mathbb{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h , that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U / (h)$. We have an exact sequence $0 \rightarrow \mathcal{O}_U \xrightarrow{h} \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is \mathcal{O}_X -privileged if and only if $h: B(K) \rightarrow B(K)$ is a split monomorphism.

Let \dot{K}_j denote the boundary of K_j , and define $\ddot{K} = \dot{K}_1 \times \dot{K}_2$ (\ddot{K} is called the Šilov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

- (i) $h: B(K) \rightarrow B(K)$ is a monomorphism.
- (i') $\exists a > 0$ such that $\|hf\| \geq a\|f\|$, $\forall f \in B(K)$.
- (ii) $X \cap \ddot{K} = \emptyset$.

(b) If $(K_1 \times K_2) \cap X = \emptyset$, then h is a split monomorphism (i.e. K is \mathcal{O}_X -privileged).

Proof: (a) (i) \Leftrightarrow (i') is a well known fact from the theory of normed vector spaces.

(ii) \Rightarrow (i'). Assume $X \cap \ddot{K} = \emptyset$. If $f \in B(K)$, then it follows from the maximum principle that $\|f\| = \sup_K |f(x)| = \sup_{\ddot{K}} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in \overset{\circ}{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $\|hf\| = \sup_K |hf(x)| \geq \geq a \sup_K |f(x)| = a \|f\|$.

(i') \Rightarrow (ii). Suppose that $X \cap \overset{\circ}{K} \neq \emptyset$ and $x = (x_1, x_2) \in X \cap \overset{\circ}{K}$. We choose an analytic function $f_1 : U_1 \rightarrow \mathbf{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \rightarrow \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \rightarrow f_1(z_1)f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K .

Applying Dini's theorem we get $\|hf^n\| \rightarrow 0$. From the inequality $a \|f^n\| \leq \|hf^n\|$ we get $\|f^n\| \rightarrow 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \rightarrow B(K)$ is a split monomorphism?

IV. FLATNESS AND PRIVILEGE

§ 1. Morphisms from an analytic space into $B(K)$

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an \mathcal{O}_S -algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S; B(K))$.

(a) Consider first $S = U' \subset \mathbf{C}^m$, U' -open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand it's obvious that ϕ is an $\mathcal{O}_{U'}$ -algebra homomorphism.

(b) Let S have a special model in the polydisc Δ in \mathbf{C}^m , defined by a sheaf \mathcal{I} of ideals of \mathcal{O}_Δ , and let \mathcal{J} be generated by f_1, \dots, f_p , V -a polycylinder neighbourhood of K in U . By Cartan's theorem B for a polycylinder,

the sequence $0 \rightarrow \mathcal{I}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \rightarrow 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K))$, $(f_1, \dots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset \subset \text{Ker } \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$, such that the diagram