

§1. Banach vector bundles over an analytic space

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that $\mathcal{O}_{W \times X}$ is \mathcal{O}_W -flat, so by scalar extension $\mathcal{O}_{S \times X}$ is \mathcal{O}_S flat.

Corollary : If X and S are two manifolds and $\pi : X \rightarrow S$ is a submersion, then π is flat.

III. PRIVILEGED POLYCYLINDERS

§ 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X .

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E . If $U \subset \mathbb{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \rightarrow E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X , i.e. X' is the support of the quotient sheaf \mathcal{O}_U/J , where $U \subset \mathbb{C}^n$ is open and J is a coherent sheaf of ideals of \mathcal{O}_U , then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$ ($V \subset U$, V -open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U .

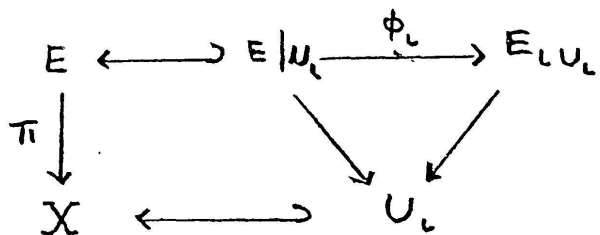
The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X , i.e. $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$, for every local model X' .

Definition 1 : The set of analytic morphisms from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F .

Definition 2 : An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi : E \rightarrow X$ a continuous projection.



Suppose that X has an open covering $(U_\iota)_{\iota \in I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E|_{U_\iota}$ and a homeomorphism ϕ_ι , such that the following diagram is commutative:

We suppose further that for each pair $\iota, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa} : E|_{U_\iota \cap U_\kappa} \rightarrow E|_{U_\iota \cap U_\kappa}$, with the underlying mapping $\phi_\iota \circ \phi_\kappa^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \quad \text{for all } \iota, \kappa, \lambda \in I.$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\phi : E \rightarrow F$ be a morphism of two Banach vector



bundles E and F , and $x \in X$.

If $\phi_x \in \mathcal{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x , such that $\phi|_U : E|_U \rightarrow F|_U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|_V = E_0|_V, F|_V = F_0|_V$ at $x \in V \subset X$ (V -open).

The set $\text{Isom}(E_0, F_0)$ of isomorphic mappings is an open subset of $\mathcal{L}(E_0, F_0)$ and the mapping $g \rightarrow g^{-1}$ is an analytic isomorphism:

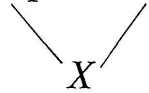
$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0).$$

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \rightarrow \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|_U)^{-1} : F|_U \rightarrow E|_U$.

Definition 3 : Let E and F be two Banach spaces and f a continuous linear mapping from E into F . f is a *split mono-(epi) morphism*, if there exists a mapping $g \in \mathcal{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definirion 4 : Let E_1 and E_2 be two Banach vector bundles over an analytic space X , and f a vector bundle morphism from E_1 into E_2 . f is a *split mono (epi) morphism*, if there exists a vector bundle morphism $g : E_2 \rightarrow E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f : E_1 \rightarrow E_2$ is a split monomorphism if and only if E_2 can



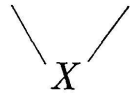
be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \rightarrow G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that } f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq E_2 \end{cases}.$$

Proposition 2 : Let $E \xrightarrow{\phi} F$ be a bundle morphism and $x \in X$.



If $\phi_x : E(x) \rightarrow F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi|_U : E|_U \rightarrow F|_U$ is a split vector bundle epi (mono) morphism.

Proof : Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|_V = E_{0V}, F|_V = F_{0V}$ at x , so that there exists a mapping $\sigma \in \mathcal{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi : F_{0V} \rightarrow E_{0V}$ by $x \rightarrow \sigma \in \mathcal{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi : F_{0V} \rightarrow E_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in x . By proposition 1 we have an isomorphic restriction $\gamma|_U, \phi|_U \circ (\psi|_U \circ (\gamma|_U)^{-1}) = I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5 : Let B_1, B_2, B_3 be Banach spaces, and $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms a *complex*, if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct

sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \rightarrow 0 \\ D_1 \simeq C_2 \end{cases} \quad k: \begin{cases} C_2 \rightarrow 0 \\ D_2 \simeq C_3 \end{cases} .$$

Definition 6: A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \\ & & X & & \end{array} \quad \text{is a complex if } g \circ f = 0.$$

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_2 \end{cases} \quad g: \begin{cases} F_2 \rightarrow 0 \\ G_2 \simeq F_3 \end{cases} .$$

Theorem 1: Let $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \end{array}$ be a complex of Banach vector

bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \xrightarrow{f_{x_0}} E_2(x_0) \xrightarrow{g_{x_0}} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0 , such that $E_1|_U \xrightarrow{f|_U} E_2|_U \xrightarrow{g|_U} E_3|_U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x , such that we have a complex $E_{1V} \xrightarrow{f|_V} E_{2V} \xrightarrow{g|_V} E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0}: \begin{cases} F_1(x_0) \rightarrow 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \quad g_{x_0}: \begin{cases} F_2(x_0) \rightarrow 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2, $f|_V : G_{1V} \rightarrow E_{2V}$, $g|_V : G_{2V} \rightarrow E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W} , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W} .$$

By our construction

$$g|_W : \begin{cases} F_2 & \rightarrow 0 \\ G_2 W & \simeq F_3 \end{cases} .$$

If $p: E_{2W} \rightarrow F_2$ is the projection with kernel G_{2W} , the map, $p \circ f: E_{1W} \rightarrow F_2$ is a split epimorphism in x_0 . Again by prop. 2 we have over an open neighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \text{Ker } p \circ f$)

$$(p \circ f)|_U : \begin{cases} F_1 & \rightarrow 0 \\ G_{1U} & \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image $f|_U(F_1)$ is contained in G_{2U} . But $g|_U \circ f|_U = 0$ and $g|_{G_{2U}}$ is a monomorphism hence $f|_U: F_1 \rightarrow 0$. We get finally (restricting all our morphisms to U)

$$f|_U : \begin{cases} F_{1U} & \rightarrow 0 \\ G_{1U} & \simeq F_{2U} \end{cases} \quad g|_U : \begin{cases} F_{2U} & \rightarrow 0 \\ G_{2U} & \xrightarrow{\sim} F_{3U} \end{cases} .$$

§ 2. Privileged polycylinders

Definition 1: A polycylinder in \mathbf{C}^n is a compact set K of the form $K = K_1 \times \dots \times K_n$ where each K_i is a compact, convex subset of \mathbf{C} , with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of \mathbf{C}^n . Let \mathcal{F} be a coherent analytic sheaf on U .

(A) There exists an open neighbourhood of K over which \mathcal{F} admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

(B) $H^q(K, \mathcal{F}) = 0$ for $q > 0$.

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf \mathcal{F} , the sequence