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*Remark* : This a particular case of the following proposition: if  $\pi$  and  $\pi'$  are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that  $\mathcal{O}_{W \times X}$  is  $\mathcal{O}_W$ -flat, so by scalar extension  $\mathcal{O}_{S \times X}$  is  $\mathcal{O}_S$  flat.

*Corollary* : If  $X$  and  $S$  are two manifolds and  $\pi : X \rightarrow S$  is a submersion, then  $\pi$  is flat.

### III. PRIVILEGED POLYCYLINDERS

#### § 1. Banach vector bundles over an analytic space

Let  $E$  be a Banach space and  $X$  an analytic space. We denote then by  $E_X$  the trivial bundle  $X \times E$  over  $X$ .

To define bundle morphisms, we first define the sheaf  $\mathcal{H}_X(E)$  of germs of analytic morphisms from  $X$  to  $E$ . If  $U \subset \mathbb{C}^n$  is open, then the set  $\mathcal{H}(U, E)$  of analytic morphisms from  $U$  into  $E$  consists of all functions  $g : U \rightarrow E$  having at every point  $x \in U$  a converging power series expansion.

Let now  $X'$  be a local model for  $X$ , i.e.  $X'$  is the support of the quotient sheaf  $\mathcal{O}_U/J$ , where  $U \subset \mathbb{C}^n$  is open and  $J$  is a coherent sheaf of ideals of  $\mathcal{O}_U$ , then  $\mathcal{H}_{X'}(E)$  is the sheaf associated to the presheaf  $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$  ( $V \subset U$ ,  $V$ -open).

*Remark* : If  $X'$  is reduced, the sections of  $\mathcal{H}_{X'}(E)$  are just the functions from  $X'$  to  $E$  which are locally induced by analytic functions on open sets in  $U$ .

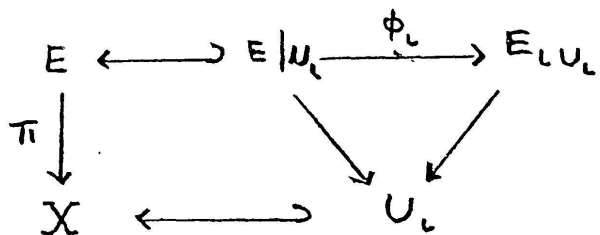
The sheaf  $\mathcal{H}_X(E)$  is constructed with help of the local models  $X'$  of  $X$ , i.e.  $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$ , for every local model  $X'$ .

*Definition 1* : The set of analytic morphisms from an analytic space  $X$  into a Banach space  $E$  is the set  $\mathcal{H}(X; E)$  of sections of the sheaf  $\mathcal{H}_X(E)$ .

Let  $\mathcal{L}(E, F)$  be the Banach space of all continuous linear mappings from the Banach space  $E$  into the Banach space  $F$ .

*Definition 2* : An analytic vector bundle morphism from  $E_X$  into  $F_X$  is an analytic morphism from  $X$  into  $\mathcal{L}(E, F)$ .

Let  $E$  be a topological space,  $X$  an analytic space, and  $\pi : E \rightarrow X$  a continuous projection.



Suppose that  $X$  has an open covering  $(U_\iota)_{\iota \in I}$ , and that for every  $\iota \in I$  there is given a trivial Banach space bundle  $E|_{U_\iota}$  and a homeomorphism  $\phi_\iota$ , such that the following diagram is commutative:

We suppose further that for each pair  $\iota, \kappa \in I$  there is given an analytic vector bundle morphism  $\gamma_{\iota\kappa} : E|_{U_\iota \cap U_\kappa} \rightarrow E|_{U_\iota \cap U_\kappa}$ , with the underlying mapping  $\phi_\iota \circ \phi_\kappa^{-1}$ , such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \quad \text{for all } \iota, \kappa, \lambda \in I.$$

This data gives a Banach vector bundle atlas on  $E$  and provides  $E$  with the structure of a Banach vector bundle over  $X$  (two atlases are equivalent if there exists an atlas containing both).

*Remark:* If  $X$  is reduced, the  $\gamma_{\iota\kappa}$  are determined by their underlying map and the condition  $\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}$  is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

*Proposition 1:* Let  $\phi : E \rightarrow F$  be a morphism of two Banach vector



bundles  $E$  and  $F$ , and  $x \in X$ .

If  $\phi_x \in \mathcal{L}(E(x), F(x))$  is an isomorphism, then there exists an open neighbourhood  $U \subset X$  of  $x$ , such that  $\phi|_U : E|_U \rightarrow F|_U$  is a vector bundle isomorphism.

*Proof:* First we take a trivialisation  $E|_V = E_0|_V, F|_V = F_0|_V$  at  $x \in V \subset X$  ( $V$ -open).

The set  $\text{Isom}(E_0, F_0)$  of isomorphic mappings is an open subset of  $\mathcal{L}(E_0, F_0)$  and the mapping  $g \rightarrow g^{-1}$  is an analytic isomorphism:

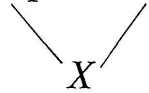
$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0).$$

So we have in an open neighbourhood  $U \subset X$  of  $x$  an analytic morphism  $y \rightarrow \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$ , which defines the inverse morphism  $(\phi|_U)^{-1} : F|_U \rightarrow E|_U$ .

*Definition 3 :* Let  $E$  and  $F$  be two Banach spaces and  $f$  a continuous linear mapping from  $E$  into  $F$ .  $f$  is a *split mono-(epi) morphism*, if there exists a mapping  $g \in \mathcal{L}(F, E)$  such that  $g \circ f = I_E$ . (Resp.  $f \circ g = I_F$ .)

*Definirion 4 :* Let  $E_1$  and  $E_2$  be two Banach vector bundles over an analytic space  $X$ , and  $f$  a vector bundle morphism from  $E_1$  into  $E_2$ .  $f$  is a *split mono (epi) morphism*, if there exists a vector bundle morphism  $g : E_2 \rightarrow E_1$  such that  $g \circ f = I_{E_1}$ . (Resp.  $f \circ g = I_{E_2}$ .)

Equivalently,  $f : E_1 \rightarrow E_2$  is a split monomorphism if and only if  $E_2$  can



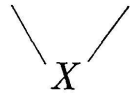
be decomposed in a direct sum  $E_2 = F_2 \oplus G_2$  such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \rightarrow G_2 \end{cases}.$$

and  $f$  is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that } f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq E_2 \end{cases}.$$

*Proposition 2 :* Let  $E \xrightarrow{\phi} F$  be a bundle morphism and  $x \in X$ .



If  $\phi_x : E(x) \rightarrow F(x)$  is a split epi (mono) morphism, then the point  $x$  has an open neighbourhood  $U \subset X$ , such that  $\phi|_U : E|_U \rightarrow F|_U$  is a split vector bundle epi (mono) morphism.

*Proof :* Suppose that  $\phi_x$  is a split epimorphism. We take first a trivilisation  $E|_V = E_{0V}$ ,  $F|_V = F_{0V}$  at  $x$ , so that there exists a mapping  $\sigma \in \mathcal{L}(F_0, E_0)$ ,  $\phi_x \circ \sigma = I_{F_0}$ . If we define a morphism  $\psi : F_{0V} \rightarrow E_{0V}$  by  $x \rightarrow \sigma \in \mathcal{L}(F_0, E_0)$ , the morphism  $\gamma = \phi \circ \psi : F_{0V} \rightarrow E_{0V}$  has an isomorphic fibre mapping  $\gamma_x = I_{F_0}$  in  $x$ . By proposition 1 we have an isomorphic restriction  $\gamma|_U$ ,  $\phi|_U \circ (\psi|_U \circ (\gamma|_U)^{-1}) = I_{F_{0U}}$ .

When  $\phi_x$  is a split monomorphism, the proof is similar.

*Definition 5 :* Let  $B_1, B_2, B_3$  be Banach spaces, and  $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$  continuous linear mappings. This sequence forms a *complex*, if  $k \circ j = 0$ . This sequence is *split exact* if the space  $B_i$  can be decomposed in direct

sums  $B_i = C_i \oplus D_i$  such that

$$j: \begin{cases} C_1 \rightarrow 0 \\ D_1 \simeq C_2 \end{cases} \quad k: \begin{cases} C_2 \rightarrow 0 \\ D_2 \simeq C_3 \end{cases} .$$

*Definition 6:* A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \\ & & X & & \end{array} \quad \text{is a complex if } g \circ f = 0.$$

The sequence is *split exact*, if every  $E_i$  can be decomposed  $E_i = F_i \oplus G_i$ , such that:

$$f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_2 \end{cases} \quad g: \begin{cases} F_2 \rightarrow 0 \\ G_2 \simeq F_3 \end{cases} .$$

*Theorem 1:* Let  $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \end{array}$  be a complex of Banach vector

bundles and  $x_0 \in X$ .

If the sequence of Banach spaces  $E_1(x_0) \xrightarrow{f_{x_0}} E_2(x_0) \xrightarrow{g_{x_0}} E_3(x_0)$  is split exact, then there exists an open neighbourhood  $U \subset X$  of  $x_0$ , such that  $E_1|_U \xrightarrow{f|_U} E_2|_U \xrightarrow{g|_U} E_3|_U$  is a split exact sequence of Banach vector bundles.

*Proof:* We take a neighbourhood  $V$  of  $x$ , such that we have a complex  $E_{1V} \xrightarrow{f|_V} E_{2V} \xrightarrow{g|_V} E_{3V}$  of trivial bundles. By assumption we have the decompositions  $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$  with

$$f_{x_0}: \begin{cases} F_1(x_0) \rightarrow 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \quad g_{x_0}: \begin{cases} F_2(x_0) \rightarrow 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2,  $f|_V : G_{1V} \rightarrow E_{2V}$ ,  $g|_V : G_{2V} \rightarrow E_{3V}$  are both split monomorphisms in a neighbourhood  $W \subset V$  of  $x_0$  and the images  $F_2 = f(G_{1W})$ ,  $F_3 = g(G_{2W})$  are subbundles of  $E_{2W}$  esp.  $E_{3W}$ , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W} .$$

By our construction

$$g|_W : \begin{cases} F_2 & \rightarrow 0 \\ G_2 W & \simeq F_3 \end{cases} .$$

If  $p: E_{2W} \rightarrow F_2$  is the projection with kernel  $G_{2W}$ , the map,  $p \circ f: E_{1W} \rightarrow F_2$  is a split epimorphism in  $x_0$ . Again by prop. 2 we have over an open neighbourhood  $U \subset W$  of  $x_0$  a decomposition  $E_{1U} = F_1 \oplus G_{1U}$  (with  $F_1 = \text{Ker } p \circ f$ )

$$(p \circ f)|_U : \begin{cases} F_1 & \rightarrow 0 \\ G_{1U} & \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image  $f|_U(F_1)$  is contained in  $G_{2U}$ . But  $g|_U \circ f|_U = 0$  and  $g|_{G_{2U}}$  is a monomorphism hence  $f|_U: F_1 \rightarrow 0$ . We get finally (restricting all our morphisms to  $U$ )

$$f|_U : \begin{cases} F_{1U} & \rightarrow 0 \\ G_{1U} & \simeq F_{2U} \end{cases} \qquad g|_U : \begin{cases} F_{2U} & \rightarrow 0 \\ G_{2U} & \xrightarrow{\sim} F_{3U} \end{cases} .$$

## § 2. Privileged polycylinders

*Definition 1:* A polycylinder in  $\mathbf{C}^n$  is a compact set  $K$  of the form  $K = K_1 \times \dots \times K_n$  where each  $K_i$  is a compact, convex subset of  $\mathbf{C}$ , with nonempty interior. If each  $K_i$  is a disc, then  $K$  is a polydisc. We first recall the following theorem of Cartan.

*Theorem 1:* Let  $K$  be a polycylinder contained in an open subset  $U$  of  $\mathbf{C}^n$ . Let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ .

(A) There exists an open neighbourhood of  $K$  over which  $\mathcal{F}$  admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

(B)  $H^q(K, \mathcal{F}) = 0$  for  $q > 0$ .

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf  $\mathcal{F}$ , the sequence