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# COMPACT ANALYTICAL VARIETIES

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## INTRODUCTION

These lectures deal with the vanishing theorem of Kodaira (cf. e.g. [2], p. 344) and some of its consequences, and with Lefschetz' theorem on hyperplane sections (cf. [1]). Only complex manifolds (and not complex spaces) are considered, but most of the results in the first part could be carried over to the more general case (with similar proofs).

### 1. PRELIMINARIES

We first give some definitions:

*Definition 1.1.* Let  $V$  be a complex manifold and  $D$  a relatively compact, open subset of  $V$ . Then  $D$  is *strongly pseudoconvex* if for every  $x_0 \in \partial D$  there exist a neighbourhood  $U$  of  $x_0$  and a real-valued  $C^2$ -function  $\varphi$  defined in  $U$  such that

$$(1) \quad d\varphi(x_0) \neq 0,$$

$$(2) \quad H(\varphi)(x_0) > 0 \text{ for all } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{C}^n - \{0\}.$$

(Here  $H(\varphi)$  is the complex Hessian form

$$\sum_{i, j=1}^n \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \alpha_i \bar{\alpha}_j$$

with respect to some system of local coordinates),

$$(3) \quad D \cap U = \{x \in U; \varphi(x) < 0\}.$$

It can be shown that strong pseudoconvexity of  $D$  is equivalent to the following property: For every  $x_0 \in \partial D$  there exist a neighbourhood  $U$  of  $x_0$  and a biholomorphic mapping  $f: U \rightarrow \Omega \subset \mathbf{C}^n$  such that  $f(U \cap D)$  has a strictly convex boundary (in the Euclidean sense).

*Definition 1.2.* Let  $V$  be a complex manifold and  $A$  a subset of  $V$ . We say that  $A$  “can be blown down to a point” if there exist an analytic space  $X$ , a point  $x_0 \in X$ , and a mapping  $f: V \rightarrow X$  such that  $f(A) = x_0$  and  $f: V - A \rightarrow X - \{x_0\}$  is an analytic isomorphism.

To give an example of sets which can be blown down to a point, we mention the following theorem (for a proof see [2], pp. 338 and 340):

*Theorem 1.3.* If  $D$  is strongly pseudoconvex, then  $D$  has a maximal compact analytic subset  $A$  whose dimension at any point is  $> 0$  and each component of  $A$  can be blown down to a point.

*Lemma 1.4.* If  $A$  can be blown down to a point, then  $A$  has a fundamental system of strongly pseudoconvex neighbourhoods.

*Proof.* Let  $X$ ,  $x_0$ , and  $f$  be as in Definition 1.2. The lemma follows from the fact that the inverse image of a strongly pseudoconvex neighbourhood of  $x_0$  is a strongly pseudoconvex neighbourhood of  $A$ .

We now introduce the concept of holomorphic line bundle.

*Definition 1.5.* Suppose  $X$  is a complex manifold. A holomorphic line bundle  $F$  on  $X$  is a complex manifold  $F$  together with a mapping  $\pi$  with the following properties:

- (i)  $\pi: F \rightarrow X$  is a holomorphic map (called projection) onto  $X$ .
- (ii) For  $x \in X$ ,  $\pi^{-1}(x)$  has the structure of a one-dimensional vector space over the complex numbers.
- (iii) For each  $x \in X$  there exist a neighbourhood  $U$  of  $x$  and a holomorphic mapping  $h$  of  $F|U = \pi^{-1}(U)$  onto  $U \times \mathbf{C}$  such that  $h^{-1}$  is holomorphic and  $h|_{\pi^{-1}(a)}$  is a  $\mathbf{C}$ -isomorphism onto  $\{a\} \times \mathbf{C}$  for every  $a \in U$ .

Let  $\{U_i\}$  be an open covering of  $X$  such that for each  $i$  we have a mapping  $h_i$  of  $F|U_i$  onto  $U_i \times \mathbf{C}$  with the properties in (iii) above. If  $U_i \cap U_j \neq \emptyset$ , we get a mapping  $h_i \circ h_j^{-1}: (U_i \cap U_j) \times \mathbf{C} \rightarrow (U_i \cap U_j) \times \mathbf{C}$ . If  $(x, c) \in (U_i \cap U_j) \times \mathbf{C}$ , then the image of  $(x, c)$  under the mapping

$h_i \circ h_j^{-1}$  can be written  $(x, \gamma'(x, c))$  where  $\gamma'(x, c) \in \mathbf{C}$ . According to the last property in (iii), for fixed  $x \in U_i \cap U_j$  the mapping  $c \rightarrow \gamma'(x, c)$  is a  $\mathbf{C}$ -isomorphism of  $\mathbf{C}$  onto itself. Therefore

$$\gamma'(x, c) = g_{ij}(x) \cdot c, \text{ where } g_{ij}(x) \neq 0, \quad (1.1)$$

and it is easily seen that  $g_{ij}$  is holomorphic in  $U_i \cap U_j$ .

The functions  $g_{ij}$  obviously satisfy the cocycle conditions

$$g_{ij} g_{jk} g_{ki} = 1 \quad \text{on} \quad U_i \cap U_j \cap U_k, \quad (1.2)$$

$$g_{ij} g_{ji} = 1 \quad \text{on} \quad U_i \cap U_j. \quad (1.3)$$

The  $g_{ij}$  are called transition functions corresponding to the line bundle  $F$ .

Conversely, it is easy to prove (cf. [4], p. 135) that given an open covering  $\{U_i\}$  and functions  $g_{ij}$  without zeros in  $U_i \cap U_j$  which satisfy the cocycle conditions, we can construct a line bundle which has  $g_{ij}$  as transition functions.

Now, let  $F$  be a line bundle over a complex manifold  $X$ , and let  $\pi$  be the corresponding projection. We denote  $\pi^{-1}(a)$  by  $F_a$ . Let  $F_a^*$  be the  $\mathbf{C}$ -dual of  $F_a$ . Then

$$F^* = \bigcup_{a \in X} F_a^*$$

is in a natural way a holomorphic line bundle over  $X$ , which is called the dual bundle of  $F$ . If  $F$  has transition functions  $\{g_{ij}\}$ , then  $F^*$  has transition functions  $\{g_{ij}^{-1}\}$ .

*Definition 1.6.* Let  $F$  be a holomorphic line bundle over a compact complex manifold. Then  $F$  is *negative* if the zero cross section  $\mathfrak{o}$  of  $F$  can be blown down to a point.  $F$  is *positive* if the dual bundle is negative.

In the sequel we let  $\underline{F}$  denote the sheaf of germs of analytic sections of a line bundle  $F$ .

## 2. THE VANISHING THEOREM OF KODAIRA

This is the following theorem, which is our first main result:

*Theorem 2.1.* Let  $X$  be a compact connected complex manifold and  $F$  a positive line bundle on  $X$  and  $S$  a coherent analytic sheaf on  $X$ . Then there exists an integer  $k(S, F)$  such that for  $k > k(S, F)$  we have  $H^q(X, S \otimes \underline{F}^k) = 0$  ( $\forall q \geq 1$ ).

The proof uses the following finiteness theorem: