

# 3. An imbedding theorem

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### 3. AN IMBEDDING THEOREM

*Lemma 3.1.* If  $X$  is a compact complex manifold and  $S$  a coherent analytic sheaf over  $X$ , then  $\Gamma(X, S)$  is a finite dimensional vector space (cf. remark concerning Theorem 2.2).

We will now prove an imbedding theorem (cf. [2], p. 343).

*Theorem 3.2.* If the complex manifold  $X$  is compact, connected, and carries a positive (negative) line bundle, then  $X$  can be imbedded biholomorphically in a complex projective space  $\mathbf{P}^N$ .

*Proof:* Suppose  $F$  is a line bundle on a compact complex manifold  $X$  with the property that for every  $a \in X$  there exists a section  $\sigma \in \Gamma(X, \underline{F})$  with  $\sigma(a) \neq 0$ . Then  $F$  defines a holomorphic mapping of  $X$  into a projective space  $\mathbf{P}^k$  in the following way:

Since  $X$  is compact,  $\Gamma(X, \underline{F})$  is finite-dimensional according to Lemma 3.1.

Let  $\sigma_0, \dots, \sigma_k$  be a basis of  $\Gamma(X, \underline{F})$ . Then the  $\sigma_j$  have no common zeros.

Since  $F$  is locally isomorphic to the product of an open subset of  $X$  and  $\mathbf{C}$ , the  $\sigma_j$  are locally given by holomorphic functions without common zeros.

We map  $X$  into  $\mathbf{P}^k$  by  $x \rightarrow (\sigma_0(x), \dots, \sigma_k(x))$ . The point in the projective space is independent of the isomorphism we are using, for if we use another isomorphism we get a point  $(g(x)\sigma_0(x), \dots, g(x)\sigma_k(x))$ , where  $g(x) \neq 0$  (cf. (1.1)).

We are now going to show that if  $F$  is positive, then there exists an integer  $\gamma$  such that the sections of  $\Gamma(X, \underline{F}^\gamma)$  have no common zeros and such that the corresponding mapping is an imbedding.

For  $a \in X$ , let  $I$  be the sheaf of germs of holomorphic functions vanishing at  $a$ . Since  $I$  is coherent, we can apply the vanishing theorem of Kodaira. We conclude that there exists an integer  $k(a)$  such that  $H^1(X, I \otimes \underline{F}^{k \geq k(a)}) = 0$

Since  $\mathcal{O}_a/I_a \approx \mathbf{C}$ , we have the following exact sequence

$$0 \rightarrow I \rightarrow \mathcal{O}(X) \rightarrow \mathbf{C}_a \rightarrow 0,$$

where  $\mathbf{C}_a$  is a sheaf with stalk  $\mathbf{C}$  at  $a$  and zero outside. From this it follows that the sequence

$$0 \rightarrow I \otimes \underline{F}^{k(a)} \rightarrow \underline{F}^{k(a)} \rightarrow \mathbf{C}_a \otimes \underline{F}^{k(a)} \rightarrow 0$$

is exact. We have  $\mathbf{C}_a \otimes \underline{F}^{k(a)} \approx \tilde{F}_a^{k(a)}$ , where  $\tilde{F}_a^{k(a)}$  has stalk  $F_a^{k(a)}$  at  $a$  and

zero outside. Using the fact that  $H^1(X, I \otimes \underline{F}^{k(a)}) = 0$ , the exact cohomology sequence associated to the above sequence of sheaves gives us an exact sequence

$$\Gamma(X, \underline{F}^{k(a)}) \rightarrow \Gamma(X, \tilde{F}_a^{k(a)}) \rightarrow 0.$$

This implies that given  $e \in F_a^{k(a)}$  there exists  $\sigma \in \Gamma(X, \underline{F}^{k(a)})$  such that  $\sigma(a) = e$ . Thus, for every  $a \in X$  we can find an integer  $k(a)$  and a neighbourhood  $V_a$  of  $a$  such that  $\Gamma(X, \underline{F}^{k(a)})$  has a section not vanishing on  $V_a$ . Since  $X$  is compact, there are finitely many such neighbourhoods  $V_i$  ( $i=1, \dots, p$ ) with corresponding sections of  $F^{k_i}$  such that  $X = \bigcup_{i=1}^p V_i$ . Letting  $k = k_1 \cdot k_2 \cdot \dots \cdot k_p$ , we get  $p$  elements of  $\Gamma(X, \underline{F}^k)$  without common zeros, for if  $\sigma \in \Gamma(X, \underline{F})$  and  $\sigma(x) \neq 0$ , then  $\sigma' = \underbrace{\sigma \otimes \dots \otimes \sigma}_{l \text{ times}} \in \Gamma(X, \underline{F}^l)$  and  $\sigma'(x) \neq 0$ .

Let  $\underline{E} = \underline{F}^k$ . Now, for  $a \in X$ , let  $G = q_a^2$ , where  $q_a$  is the ideal of germs of holomorphic functions vanishing at  $a$ . Using the above argument with  $\underline{E}$  and  $G$  instead of  $\underline{F}$  and  $I$ , we see that there exists an integer  $s(a)$  such that the restriction mapping

$$\Gamma(X, \underline{E}^{s(a)}) \rightarrow \{ \mathcal{O}_a / q_a^2 \} \otimes \underline{E}_a^{s(a)}$$

is surjective. Since the residue classes in  $\mathcal{O}_a / q_a^2$  are sets of germs  $f$  of holomorphic functions at  $a$  with fixed values of  $f(a)$  and  $df(a)$ , this implies that we can find a neighbourhood  $U_a$  of  $a$  and sections  $\sigma_1, \dots, \sigma_t \in \Gamma(X, \underline{E}^{s(a)})$  which are nowhere zero in  $U_a$  such that the mapping given by  $\sigma_1, \dots, \sigma_t$  is regular and injective in  $U_a$ . We observe that for every positive integer  $l$  we can find sections  $\sigma_1^{(1)}, \dots, \sigma_t^{(1)} \in \Gamma(X, \underline{E}^{ls(a)})$  which have the same properties in  $U_a$ . In fact, if  $\sigma$  is a section of  $\underline{E}^{s(a)}$  which has no zeros on a set  $M \subset X$ , we set

$$\sigma' = \sigma \otimes \dots \otimes \sigma, (l-1) \text{ times.}$$

Then  $\sigma' \otimes \sigma_1, \dots, \sigma' \otimes \sigma_t$  are sections of  $\underline{E}^{ls(a)}$ , and define the same mapping (at least on  $M$ ) as  $\sigma_1, \dots, \sigma_t$ .

We can cover  $X$  by finitely many such neighbourhoods  $U_1, \dots, U_r$ . If  $s' = s_1 \cdot \dots \cdot s_r$ , then there are elements of  $\Gamma(X, \underline{E}^{s'})$  which give a regular, injective mapping in each  $U_i$  ( $1 \leq i \leq r$ ).

We are now going to show that we can separate points in  $X$  by sections of a suitable  $\underline{E}^z$ . Let  $U = \bigcup_{i=1}^r (U_i \times U_i)$ . For  $(a, b) \in X \times X - U$ , let  $H$  be the sheaf of germs of holomorphic functions vanishing at  $a$  and  $b$ . It is

easily seen that the sequence

$$0 \rightarrow H \rightarrow \mathcal{O}(X) \rightarrow \mathbf{C}_a \oplus \mathbf{C}_b \rightarrow 0$$

is exact. From this we conclude as above that there exists an integer  $s(a, b)$  such that the sequence

$$\Gamma(X, \underline{E}^{s(a,b)}) \rightarrow E_a^{s(a,b)} \oplus E_b^{s(a,b)} \rightarrow 0$$

is exact. Therefore there exists a neighbourhood  $W$  of  $(a, b)$  in  $X \times X$  such that if  $(a', b') \in W$ , then the sections of  $\Gamma(X, \underline{E}^{s(a,b)})$  separate  $a'$  and  $b'$ ; that is, if  $\sigma_0, \dots, \sigma_k$  is a basis of  $\Gamma(X, \underline{E}^{s(a,b)})$ , then  $(\sigma_0(a'), \dots, \sigma_k(a'))$  and  $(\sigma_0(b'), \dots, \sigma_k(b'))$  are different points in  $\mathbf{P}^k$ . Let  $l$  be a positive integer, let  $(a', b') \in W$ , and let  $\sigma$  be a section of  $\Gamma(X, \underline{E}^{s(a,b)})$  such that  $\sigma(a') \neq 0$  and  $\sigma(b') \neq 0$ . Then  $\sigma^{l-1} \otimes \sigma_0, \dots, \sigma^{l-1} \otimes \sigma_k$  are sections of  $\Gamma(X, \underline{E}^{ls(a,b)})$  such that  $((\sigma^{l-1} \otimes \sigma_0)(a'), \dots, (\sigma^{l-1} \otimes \sigma_k)(a'))$  and  $((\sigma^{l-1} \otimes \sigma_0)(b'), \dots, (\sigma^{l-1} \otimes \sigma_k)(b'))$  are different points in  $\mathbf{P}^k$ .

This means that for every positive integer  $l$  the sections of  $\Gamma(X, \underline{E}^{ls(a,b)})$  separate all point pairs in  $W$ . Thus, covering  $X \times X - U$  by finitely many such neighbourhoods and taking  $s''$  to be the product of the corresponding  $s(a, b)$ , we find that the sections of  $\Gamma(X, \underline{E}^{s''})$  separate all point pairs in  $X \times X - U$ .

Let  $\alpha = s's''$  and let  $\sigma_0, \dots, \sigma_d$  be a basis of  $\Gamma(X, \underline{E}^\alpha)$ . We claim that the mapping  $f$  from  $X$  into  $\mathbf{P}^d$  defined by  $f(x) = (\sigma_0(x), \dots, \sigma_d(x))$  is a biholomorphic imbedding of  $X$  into  $\mathbf{P}^d$ . That this mapping is regular follows from the fact that  $\alpha$  is a multiple of  $s'$ . What remains to be proved is that the mapping is injective.

Suppose  $a, b \in X$ ,  $a \neq b$ . If  $(a, b) \in U$ , then  $a, b \in U_i$  for some  $i$ , and since  $\alpha$  is a multiple of  $s'$ , we have  $f(a) \neq f(b)$ . If  $(a, b) \in X \times X - U$ , then  $f(a) \neq f(b)$  since  $\alpha$  is a multiple of  $s''$ . This proves the theorem.

#### 4. LINE BUNDLE ASSOCIATED TO A DIVISOR

Let  $X$  be a complex manifold and  $D$  an analytic subset of  $X$  of pure codimension 1 at every point. Such a set  $D$  is called a *divisor* of  $X$ . We shall construct a line bundle  $F$  on  $X$ , associated to  $D$ .

To do this, we observe that every point of  $X$  has a neighbourhood  $U$  in which there is a holomorphic function  $s$  such that  $U \cap D = \{x \in U; s(x) = 0\}$ , and  $s$  generates, at every point of  $U$ , the ideal of germs of holomorphic functions vanishing on  $D$ . Thus we get a covering of  $X$  by open sets  $U_j$  and