5. Meromorphic forms

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

on A. Since $f \in H^0(\mathbf{P}^n, F^m)$, this gives the desired homogeneous polynomial.

To prove the theorem, it now suffices to consider all homogeneous polynomials which vanish on A without being identically zero and apply the Hilbert basis theorem.

5. Meromorphic forms

Let X be a complex manifold. A holomorphic differential form is a form which in local coordinates can be written as a finite sum

$$\omega = \Sigma a_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k} \tag{5.1}$$

with holomorphic coefficients $a_{i_1 \dots i_k}$.

A form is called meromorphic if it has locally the form (5.1) with coefficients that are meromorphic functions. Every meromorphic function can be written locally as $f\omega$ where f is a meromorphic function and ω a holomorphic form. The exterior differentiation d, satisfying $d^2 = 0$, extends naturally to meromorphic forms.

Let D be a divisor of X and let $\Omega^p(k, D) = \Omega^p(X, k, D)$ be the sheaf of germs of meromorphic p-forms on X with poles only on D and of order $\leqslant k$, and let $\Omega^p = \Omega^p(X)$ be the sheaf of germs of holomorphic p-forms on X.

Lemma 5.1. There is a natural isomorphism

$$\Omega^p(k, D) \simeq \Omega^p \otimes \underline{F}^k .$$

Proof. A germ in $\Omega^p(k, D)$ at $a \in X$ is represented by a form $f\omega$, where f is a meromorphic function in a neighbourhood U of a, with poles only on D and of order $\leqslant k$, and ω is a holomorphic form on U. Now to f corresponds biuniquely a section $s \in \Gamma(U, F^k)$ (see Sect. 4), which gives a germ $s_a \in \underline{F}^k_a$. Also ω defines a germ $\omega_a \in \Omega^p_a$.

The desired mapping $\Omega^p(K, D) \to \Omega^p \otimes \underline{F}^k$ is now uniquely defined by

$$f\omega \to \omega_a \otimes s_a .$$

To see that it is an isomorphism, it is sufficient to observe that the inverse mapping of $\Omega^p \otimes \underline{F}^k$ into $\Omega^p(k, D)$ is induced by the bilinear mapping $\Omega^p \oplus \underline{F}^k \to \Omega^p(k, D)$, which is given by

$$(\omega_a, s_a) \to (f\omega)_a , \qquad (a \in X) .$$

where f is the meromorphic function determined by s_a by the procedure described just before Th. 4.1.

Now let X be a compact submanifold of $\mathbf{P}^n$ and consider hyperplanes H_c in $\mathbf{P}^n$, given in homogeneous coordinates $z_0, \ldots, z_n$ by equations

$$\sum_0^n c_j z_j = 0 \text{ where } c = (c_0, \ldots, c_n) \neq 0 .$$

Theorem 5.2. There is an open dense set Ω in $\mathbf{C}^{n+1}$ such that if $c = (c_0, \ldots, c_n) \in \Omega$, the hyperplane section $D_c = H_c \cap X$ is a non-singular analytic subset of X.

The proof is omitted here.

Let $D = H \cap X$ be a non-singular hyperplane section of X. To D is then associated a positive line bundle F on X (see Sect. 4). By Kodaira's vanishing theorem there is a k_0 such that

$$H^q(X, \Omega^p \otimes \underline{F}^k) = 0 , \qquad (\forall q \geq 1, \forall k \geqq k_0) .$$

Using the isomorphism in Lemma 5.1, we have therefore proved.

Lemma 5.3. If D is a non-singular hyperplane section of a compact submanifold X of $\mathbf{P}^n$, then there exists k_0 such that

$$H^q(X, \Omega^p(k, D)) = 0 , \qquad (\forall q \geq 1, \forall k \geq k_0) .$$

6. The Atiyah-Hodge theorem

We first recall two well-known theorems.

Let X be a paracompact C^∞ manifold and let $\mathscr{E}^p$ be the sheaf of germs of C^∞ p-forms on X ($p=0, 1, \ldots$).

Then the sequence

$$0 \to \mathbf{C} \xrightarrow{i} \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \xrightarrow{d} \cdots \tag{6.1}$$

is exact (Poincaré's lemma), and

$$H^q(X, \mathscr{E}^p) = 0 , \qquad (\forall q \geq 1, \forall p \geqq 0) , \tag{6.2}$$

because the $\mathscr{E}^p$ are fine sheaves, i.e. they have partitions of unity. From (6.1) we get the sequence

$$0 \to \Gamma(X, \mathscr{E}^0) \to \Gamma(X, \mathscr{E}^1) \to \ldots ,$$

which need not be exact. Put