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THE COHERENCE OF DIRECT IMAGES

by H. Grauert

Introduction

The coherence of the direct images of coherent sheaves was treated in the paper [1]: H. Grauert: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen (*Pub. Math.* IHES 1960, pp. 5-64, corrections 1963). This paper deals with the most general case and its technique is very difficult. The main point in the proof is the Hauptlemma on page 47. Here a proof of this Hauptlemma in the case of regular families of compact complex manifolds and locally free analytic sheaves is given. Although this special case is easier than the general, the ideas are practically the same. Therefore these lecture notes of some talks given by H. Grauert, Helsinki 1967, may lead to an understanding of the general proof. In these notes only the Hauptlemma is proved. The proof of coherence is omitted. This part is more formal and can be done like in [1] on p. 55. See [1] for applications of the theorem.

A detailed presentation of the proof in the general case is given also by Knorr [2].

Cohomology theory

In this paper we use Čech cohomology. We shall briefly show how this cohomology is defined. In the following discussion X denotes a connected complex analytic manifold, $\mathcal{O}$ is the sheaf of germs of holomorphic functions and S a sheaf of $\mathcal{O}$-modules. Let $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ be an open covering of X. We put $U_{\iota_0 \dots \iota_l} = U_{\iota_0} \cap \dots \cap U_{\iota_l}$. We consider cochains of order l with values in S. Let us put $C^l(\mathfrak{U}, S) = \{ \xi \}$ where ξ denotes a full collection of crossections $\xi_{\iota_0 \dots \iota_l}$ over all $U_{\iota_0 \dots \iota_l}$. We always assume that $\xi_{\iota_0 \dots \iota_l}$ is anticommutative in its indices. In the system $\{ C^l(\mathfrak{U}, S) \}$ we have the usual coboundary map $\delta : C^l(\mathfrak{U}, S) \to C^{l+1}(\mathfrak{U}, S)$ which makes the system a complex. We put $Z^l(\mathfrak{U}, S) = \text{Ker}\, \delta \subset C^l(\mathfrak{U}, S)$ and $B^l(\mathfrak{U}, S) = \delta(C^{l-1}(\mathfrak{U}, S))$. The $l-$ th cohomology group $H^l(\mathfrak{U}, S)$ with respect to the open covering $\mathfrak{U}$ is $Z^l(\mathfrak{U}, S)/B^l(\mathfrak{U}, S)$. An open covering $\mathfrak{V} = \{V_\nu\}_{\nu \in N}$ is finer than an open covering $\mathfrak{U} = \{U_\iota\}_{\iota \in J}$ if there exists an index map $\tau : N \to J$ such that $V_\nu \subset U_{\tau(\nu)}$ for $\nu \in N$. It follows that an element of $\Gamma(U_{\tau(\nu_0)} \dots {}_{\tau(\nu_l)}, S)$ can be restricted to a continuous crossection over $V_{\nu_0 \dots \nu_l}$. In this way we get a map $\tau^* : C^l(\mathfrak{U}, S) \to C^l(\mathfrak{V}, S)$. The following diagram is commutative:

$$\begin{array}{ccc} C^l(\mathfrak{U}, S) & \overset{\tau^*}{\to} & C^l(\mathfrak{V}, S) \\ \delta\downarrow & & \delta\downarrow \\ C^{l+1}(\mathfrak{U}, S) & \overset{\tau^*}{\to} & C^{l+1}(\mathfrak{V}, S) \end{array}$$

It follows that we have a map $\tau^* : Z^l(\mathfrak{U}, S) \to Z^l(\mathfrak{V}, S)$. Let us put $Z^l(X, S) = \bigcup_{\mathfrak{U}} Z^l(\mathfrak{U}, S)$, where $\mathfrak{U}$ runs over all open coverings of X. In $Z^l(X, S)$ we can introduce an equivalence relation $\approx$ as follows: Let $\xi_1 \in Z^l(\mathfrak{U}, S)$ and $\xi_2 \in Z^l(\mathfrak{U}_1, S)$. We put $\xi_1 \approx \xi_2$ iff there exists $\mathfrak{U}_2$ such that $\mathfrak{U}_2$ is finer than $\mathfrak{U}$ and $\mathfrak{U}_1$ and $\xi_1|\mathfrak{U}_2 - \xi_2|\mathfrak{U}_2 \in B^l(\mathfrak{U}_2, S)$. Here we have put $\xi_\nu|\mathfrak{U}_2 = \tau_\nu^*(\xi_\nu)$ where τ_ν^* comes from an index map $\tau_\nu : \mathfrak{U}_2 \to \mathfrak{U}, \mathfrak{U}_1$. It is easy to check that the equivalence relation defined on $Z^l(X, S)$ is independent of the index maps. Now $H^l(X, S)$ is the set of equivalence classes in $Z^l(X, S)$. Because $C^l(\mathfrak{U}, S)$ is a module over the ring $I(X)$ of holomorphic functions on X it follows that $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ are modules over $I(X)$. We have a natural homomorphism $H^l(\mathfrak{U}, S) \to H^l(X, S)$. Let now $X' \subset X$ be an open subset. Then X' is a complex analytic manifold. We put $S' = S|X'$ and $\mathfrak{U}' = \mathfrak{U} \cap X' = \{U_\iota \cap X'\}$ and obtain an open covering of X'. The restriction of crossections gives a homomorphism $\gamma : C^l(\mathfrak{U}, S) \to C^l(\mathfrak{U}', S')$ which commutes with δ and any index map τ. Thus we obtain restriction homomorphisms: $H^l(\mathfrak{U}, S) \to H^l(\mathfrak{U}', S')$ and $H^l(X, S) \to H^l(X', S')$.

STEIN MANIFOLDS

A complex analytic manifold X is a Stein manifold if: 1) X is holomorphically convex, i.e. if $D = (x_\nu)_1^\infty$ is an infinite discrete set, then there exists $f \in I(X)$ such that $|f(D)| = \sup_\nu |f(x_\nu)|$ is infinite. 2) X can be