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usual coboundary map $\delta : C^l(\mathfrak{U}, S) \to C^{l+1}(\mathfrak{U}, S)$ which makes the system a complex. We put $Z^l(\mathfrak{U}, S) = \operatorname{Ker} \delta \subset C^l(\mathfrak{U}, S)$ and $B^l(\mathfrak{U}, S) = \delta(C^{l-1}(\mathfrak{U}, S))$. The $l-$ th cohomology group $H^l(\mathfrak{U}, S)$ with respect to the open covering $\mathfrak{U}$ is $Z^l(\mathfrak{U}, S)/B^l(\mathfrak{U}, S)$. An open covering $\mathfrak{V} = \{ V_\nu \}_{\nu \in N}$ is finer than an open covering $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ if there exists an index map $\tau : N \to J$ such that $V_\nu \subset U_{\tau(\nu)}$ for $\nu \in N$. It follows that an element of $\Gamma(U_{\tau(\nu_0)} \dots {}_{\tau(\nu_l)}, S)$ can be restricted to a continuous crossection over $V_{\nu_0 \dots \nu_l}$. In this way we get a map $\tau^* : C^l(\mathfrak{U}, S) \to C^l(\mathfrak{V}, S)$. The following diagram is commutative:

$$\begin{array}{ccc} C^l(\mathfrak{U}, S) & \xrightarrow{\tau^*} & C^l(\mathfrak{V}, S) \\ \delta\downarrow & & \delta\downarrow \\ C^{l+1}(\mathfrak{U}, S) & \xrightarrow{\tau^*} & C^{l+1}(\mathfrak{V}, S) \end{array}$$

It follows that we have a map $\tau^* : Z^l(\mathfrak{U}, S) \to Z^l(\mathfrak{V}, S)$. Let us put $Z^l(X, S) = \bigcup\limits_{\mathfrak{U}} Z^l(\mathfrak{U}, S)$, where $\mathfrak{U}$ runs over all open coverings of X. In $Z^l(X, S)$ we can introduce an equivalence relation $\approx$ as follows: Let $\xi_1 \in Z^l(\mathfrak{U}, S)$ and $\xi_2 \in Z^l(\mathfrak{U}_1, S)$. We put $\xi_1 \approx \xi_2$ iff there exists $\mathfrak{U}_2$ such that $\mathfrak{U}_2$ is finer than $\mathfrak{U}$ and $\mathfrak{U}_1$ and $\xi_1|\mathfrak{U}_2 - \xi_2|\mathfrak{U}_2 \in B^l(\mathfrak{U}_2, S)$. Here we have put $\xi_\nu|\mathfrak{U}_2 = \tau_\nu^*(\xi_\nu)$ where τ_ν^* comes from an index map $\tau_\nu : \mathfrak{U}_2 \to \mathfrak{U}, \mathfrak{U}_1$. It is easy to check that the equivalence relation defined on $Z^l(X, S)$ is independent of the index maps. Now $H^l(X, S)$ is the set of equivalence classes in $Z^l(X, S)$. Because $C^l(\mathfrak{U}, S)$ is a module over the ring $I(X)$ of holomorphic functions on X it follows that $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ are modules over $I(X)$. We have a natural homomorphism $H^l(\mathfrak{U}, S) \to H^l(X, S)$. Let now $X' \subset X$ be an open subset. Then X' is a complex analytic manifold. We put $S' = S|X'$ and $\mathfrak{U}' = \mathfrak{U} \cap X' = \{ U_\iota \cap X' \}$ and obtain an open covering of X'. The restriction of crossections gives a homomorphism $\gamma : C^l(\mathfrak{U}, S) \to C^l(\mathfrak{U}', S')$ which commutes with δ and any index map τ. Thus we obtain restriction homomorphisms: $H^l(\mathfrak{U}, S) \to H^l(\mathfrak{U}', S')$ and $H^l(X, S) \to H^l(X', S')$.

Stein manifolds

A complex analytic manifold X is a Stein manifold if: 1) X is holomorphically convex, i.e. if $D = (x_\nu)_1^\infty$ is an infinite discrete set, then there exists $f \in I(X)$ such that $|f(D)| = \sup\limits_\nu |f(x_\nu)|$ is infinite. 2) X can be spread holomorphically, i.e. for any $x \in X$ there exists $f_1 \dots f_N \in I(X)$ such that x is an isolated common zero of $f_1 \dots f_N$.

Let X be a complex analytic manifold. A Stein covering $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ of X is an open covering of X such that every U_ι is Stein. We shall often use the following result:

Leray's Theorem: If $\mathfrak{U}$ is a Stein covering of X then $H^l(\mathfrak{U}, S) \to H^l(X, S)$ is an isomorphism for every coherent analytic sheaf S.

The isomorphism between $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ means the following: If $\underline{\xi} \in H^l(X, S)$ there exists $\xi \in Z^l(\mathfrak{U}, S)$ such that ξ maps into $\underline{\xi}$ under the natural homomorphism $Z^l(\mathfrak{U}, S) \to H^l(X, S)$ and moreover if $\xi \in Z^l(\mathfrak{U}, S)$ is mapped into zero in $H^l(X, S)$ there exist $\eta \in C^{l-1}(\mathfrak{U}, S)$ such that $\xi = \delta\eta$ in $C^l(\mathfrak{U}, S)$.

Direct images of sheaves

Let X and Y be complex analytic manifolds. Let $\psi : X \to Y$ be a holomorphic map and let S be an analytic sheaf on X. Now X is fibered by the fibers $X(y) = \psi^{-1}(y)$ for $y \in Y$. Let U be an open neighborhood of a point $y \in Y$, then $V = \psi^{-1}(U)$ is an open set in X. Hence V is a complex analytic manifold and the restriction of S to V gives an analytic sheaf on V. We can now define $H^l(V, S)$. Let us put $H_y^l = \bigcup_U H^l(\psi^{-1}(U), S)$ where U runs over all open neighborhoods of y in Y. In H_y^l we introduce an equivalence relation as follows: $\xi_1 \in H^l(\psi^{-1}(U_1), S)$ and $\xi_2 \in H^l(\psi^{-1}(U_2), S)$ are equivalent iff there exists $U = U(y)$ in Y such that $U \subset U_1 \cap U_2$ and $\xi_1|\psi^{-1}(U) = \xi_2|\psi^{-1}(U)$ in $H^l(\psi^{-1}(U), S)$. We let $\psi_{(l)}(S)_{(y)}$ denote the set of equivalence classes in H_y^l. The equivalence class generated by $\xi \in H^l(\psi^{-1}(U), S)$ is denoted by ξ_y. The set $\psi_{(l)}(S)_{(y)}$ is called the set of germs of cohomology classes of dimension l along the fiber $X(y)$. Now $\psi_{(l)}(S)_{(y)}$ is an $\mathcal{O}_{y,Y}$-module. For if $g_y \in \mathcal{O}_{y,Y}$ we have a representative $g \in I(U)$ for some open neighborhood U of y. Then $g \circ \psi \in$ $\in I(\psi^{-1}(U))$. If $\xi_y \in \psi_{(l)}(S)_{(y)}$ and U is sufficiently small we can find a representative $\xi \in H^l(\psi^{-1}(U), S)$ for ξ_y. Then we put $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$. Now we form $\psi_{(l)}(S) = \bigcup_{y \in Y} \psi_{(l)}(S)_{(y)}$ where we introduce a sheaf topology. A base of the open sets are $\{ \xi_y : y \in U \}$ for $\xi \in H^l(\psi^{-1}(U), S)$. If $\xi \in H^l(X, S)$ then the map $y \to \xi_y$ is a cross-section in $\psi_{(l)}(S)$. We call it the direct image of ξ and denote it by $\psi_{(l)}(\xi)$. The sheaf $\psi_{(l)}(S)$ is the direct