

# Stein manifolds

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usual coboundary map  $\delta : C^l(\mathfrak{U}, S) \rightarrow C^{l+1}(\mathfrak{U}, S)$  which makes the system a complex. We put  $Z^l(\mathfrak{U}, S) = \text{Ker } \delta \subset C^l(\mathfrak{U}, S)$  and  $B^l(\mathfrak{U}, S) = \delta(C^{l-1}(\mathfrak{U}, S))$ . The  $l$ -th cohomology group  $H^l(\mathfrak{U}, S)$  with respect to the open covering  $\mathfrak{U}$  is  $Z^l(\mathfrak{U}, S)/B^l(\mathfrak{U}, S)$ . An open covering  $\mathfrak{B} = \{V_\nu\}_{\nu \in N}$  is finer than an open covering  $\mathfrak{U} = \{U_i\}_{i \in J}$  if there exists an index map  $\tau : N \rightarrow J$  such that  $V_\nu \subset U_{\tau(\nu)}$  for  $\nu \in N$ . It follows that an element of  $\Gamma(U_{\tau(\nu_0)} \dots \tau(\nu_l), S)$  can be restricted to a continuous crosssection over  $V_{\nu_0} \dots \nu_l$ . In this way we get a map  $\tau^* : C^l(\mathfrak{U}, S) \rightarrow C^l(\mathfrak{B}, S)$ . The following diagram is commutative:

$$\begin{array}{ccc} & \tau^* & \\ C^l(\mathfrak{U}, S) & \rightarrow & C^l(\mathfrak{B}, S) \\ \delta \downarrow & & \delta \downarrow \\ & \tau^* & \\ C^{l+1}(\mathfrak{U}, S) & \rightarrow & C^{l+1}(\mathfrak{B}, S) \end{array}$$

It follows that we have a map  $\tau^* : Z^l(\mathfrak{U}, S) \rightarrow Z^l(\mathfrak{B}, S)$ . Let us put  $Z^l(X, S) = \bigcup_{\mathfrak{U}} Z^l(\mathfrak{U}, S)$ , where  $\mathfrak{U}$  runs over all open coverings of  $X$ . In

$Z^l(X, S)$  we can introduce an equivalence relation  $\approx$  as follows: Let  $\xi_1 \in Z^l(\mathfrak{U}, S)$  and  $\xi_2 \in Z^l(\mathfrak{U}_1, S)$ . We put  $\xi_1 \approx \xi_2$  iff there exists  $\mathfrak{U}_2$  such that  $\mathfrak{U}_2$  is finer than  $\mathfrak{U}$  and  $\mathfrak{U}_1$  and  $\xi_1|_{\mathfrak{U}_2} - \xi_2|_{\mathfrak{U}_2} \in B^l(\mathfrak{U}_2, S)$ . Here we have put  $\xi_\nu|_{\mathfrak{U}_2} = \tau_\nu^*(\xi_\nu)$  where  $\tau_\nu^*$  comes from an index map  $\tau_\nu : \mathfrak{U}_2 \rightarrow \mathfrak{U}, \mathfrak{U}_1$ . It is easy to check that the equivalence relation defined on  $Z^l(X, S)$  is independent of the index maps. Now  $H^l(X, S)$  is the set of equivalence classes in  $Z^l(X, S)$ . Because  $C^l(\mathfrak{U}, S)$  is a module over the ring  $I(X)$  of holomorphic functions on  $X$  it follows that  $H^l(\mathfrak{U}, S)$  and  $H^l(X, S)$  are modules over  $I(X)$ . We have a natural homomorphism  $H^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$ . Let now  $X' \subset X$  be an open subset. Then  $X'$  is a complex analytic manifold. We put  $S' = S|_{X'}$  and  $\mathfrak{U}' = \mathfrak{U} \cap X' = \{U_i \cap X'\}$  and obtain an open covering of  $X'$ . The restriction of crosssections gives a homomorphism  $\gamma : C^l(\mathfrak{U}, S) \rightarrow C^l(\mathfrak{U}', S')$  which commutes with  $\delta$  and any index map  $\tau$ . Thus we obtain restriction homomorphisms:  $H^l(\mathfrak{U}, S) \rightarrow H^l(\mathfrak{U}', S')$  and  $H^l(X, S) \rightarrow H^l(X', S')$ .

#### STEIN MANIFOLDS

A complex analytic manifold  $X$  is a Stein manifold if: 1)  $X$  is holomorphically convex, i.e. if  $D = (x_\nu)_1^\infty$  is an infinite discrete set, then there exists  $f \in I(X)$  such that  $|f(D)| = \sup_\nu |f(x_\nu)|$  is infinite. 2)  $X$  can be

spread holomorphically, i.e. for any  $x \in X$  there exists  $f_1 \dots f_N \in I(X)$  such that  $x$  is an isolated common zero of  $f_1 \dots f_N$ .

Let  $X$  be a complex analytic manifold. A Stein covering  $\mathfrak{U} = \{U_i\}_{i \in J}$  of  $X$  is an open covering of  $X$  such that every  $U_i$  is Stein. We shall often use the following result:

*Leray's Theorem:* If  $\mathfrak{U}$  is a Stein covering of  $X$  then  $H^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$  is an isomorphism for every coherent analytic sheaf  $S$ .

The isomorphism between  $H^l(\mathfrak{U}, S)$  and  $H^l(X, S)$  means the following: If  $\underline{\xi} \in H^l(X, S)$  there exists  $\xi \in Z^l(\mathfrak{U}, S)$  such that  $\xi$  maps into  $\underline{\xi}$  under the natural homomorphism  $Z^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$  and moreover if  $\underline{\xi} \in Z^l(\mathfrak{U}, S)$  is mapped into zero in  $H^l(X, S)$  there exist  $\eta \in C^{l-1}(\mathfrak{U}, S)$  such that  $\xi = \delta\eta$  in  $C^l(\mathfrak{U}, S)$ .

#### DIRECT IMAGES OF SHEAVES

Let  $X$  and  $Y$  be complex analytic manifolds. Let  $\psi : X \rightarrow Y$  be a holomorphic map and let  $S$  be an analytic sheaf on  $X$ . Now  $X$  is fibered by the fibers  $X(y) = \psi^{-1}(y)$  for  $y \in Y$ . Let  $U$  be an open neighborhood of a point  $y \in Y$ , then  $V = \psi^{-1}(U)$  is an open set in  $X$ . Hence  $V$  is a complex analytic manifold and the restriction of  $S$  to  $V$  gives an analytic sheaf on  $V$ . We can now define  $H^l(V, S)$ . Let us put  $H_y^l = \bigcup_U H^l(\psi^{-1}(U), S)$

where  $U$  runs over all open neighborhoods of  $y$  in  $Y$ . In  $H_y^l$  we introduce an equivalence relation as follows:  $\xi_1 \in H^l(\psi^{-1}(U_1), S)$  and  $\xi_2 \in H^l(\psi^{-1}(U_2), S)$  are equivalent iff there exists  $U = U(y)$  in  $Y$  such that  $U \subset U_1 \cap U_2$  and  $\xi_1|_{\psi^{-1}(U)} = \xi_2|_{\psi^{-1}(U)}$  in  $H^l(\psi^{-1}(U), S)$ . We let  $\psi_{(l)}(S)_{(y)}$  denote the set of equivalence classes in  $H_y^l$ . The equivalence class generated by  $\xi \in H^l(\psi^{-1}(U), S)$  is denoted by  $\xi_y$ . The set  $\psi_{(l)}(S)_{(y)}$  is called the set of germs of cohomology classes of dimension  $l$  along the fiber  $X(y)$ . Now  $\psi_{(l)}(S)_{(y)}$  is an  $\mathcal{O}_{y,Y}$ -module. For if  $g_y \in \mathcal{O}_{y,Y}$  we have a representative  $g \in I(U)$  for some open neighborhood  $U$  of  $y$ . Then  $g \circ \psi \in I(\psi^{-1}(U))$ . If  $\xi_y \in \psi_{(l)}(S)_{(y)}$  and  $U$  is sufficiently small we can find a representative  $\xi \in H^l(\psi^{-1}(U), S)$  for  $\xi_y$ . Then we put  $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$ . Now we form  $\psi_{(l)}(S) = \bigcup_{y \in Y} \psi_{(l)}(S)_{(y)}$  where we introduce a sheaf topology.

A base of the open sets are  $\{\xi_y : y \in U\}$  for  $\xi \in H^l(\psi^{-1}(U), S)$ . If  $\xi \in H^l(X, S)$  then the map  $y \rightarrow \xi_y$  is a cross-section in  $\psi_{(l)}(S)$ . We call it the direct image of  $\xi$  and denote it by  $\psi_{(l)}(\xi)$ . The sheaf  $\psi_{(l)}(S)$  is the direct