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spread holomorphically, i.e. for any $x \in X$ there exists $f_1 \dots f_N \in I(X)$ such that $x$ is an isolated common zero of $f_1 \dots f_N$.

Let $X$ be a complex analytic manifold. A Stein covering $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ of $X$ is an open covering of $X$ such that every $U_\iota$ is Stein. We shall often use the following result:

*Leray's Theorem:* If $\mathfrak{U}$ is a Stein covering of $X$ then $H^l(\mathfrak{U}, S) \to H^l(X, S)$ is an isomorphism for every coherent analytic sheaf $S$.

The isomorphism between $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ means the following: If $\underline{\xi} \in H^l(X, S)$ there exists $\xi \in Z^l(\mathfrak{U}, S)$ such that $\xi$ maps into $\underline{\xi}$ under the natural homomorphism $Z^l(\mathfrak{U}, S) \to H^l(X, S)$ and moreover if $\xi \in Z^l(\mathfrak{U}, S)$ is mapped into zero in $H^l(X, S)$ there exist $\eta \in C^{l-1}(\mathfrak{U}, S)$ such that $\xi = \delta\eta$ in $C^l(\mathfrak{U}, S)$.

## Direct images of sheaves

Let $X$ and $Y$ be complex analytic manifolds. Let $\psi : X \to Y$ be a holomorphic map and let $S$ be an analytic sheaf on $X$. Now $X$ is fibered by the fibers $X(y) = \psi^{-1}(y)$ for $y \in Y$. Let $U$ be an open neighborhood of a point $y \in Y$, then $V = \psi^{-1}(U)$ is an open set in $X$. Hence $V$ is a complex analytic manifold and the restriction of $S$ to $V$ gives an analytic sheaf on $V$. We can now define $H^l(V, S)$. Let us put $H^l_y = \bigcup_U H^l(\psi^{-1}(U), S)$ where $U$ runs over all open neighborhoods of $y$ in $Y$. In $H^l_y$ we introduce an equivalence relation as follows: $\xi_1 \in H^l(\psi^{-1}(U_1), S)$ and $\xi_2 \in H^l(\psi^{-1}(U_2), S)$ are equivalent iff there exists $U = U(y)$ in $Y$ such that $U \subset U_1 \cap U_2$ and $\xi_1 | \psi^{-1}(U) = \xi_2 | \psi^{-1}(U)$ in $H^l(\psi^{-1}(U), S)$. We let $\psi_{(l)}(S)_{(y)}$ denote the set of equivalence classes in $H^l_y$. The equivalence class generated by $\xi \in H^l(\psi^{-1}(U), S)$ is denoted by $\xi_y$. The set $\psi_{(l)}(S)_{(y)}$ is called the set of germs of cohomology classes of dimension $l$ along the fiber $X(y)$. Now $\psi_{(l)}(S)_{(y)}$ is an $\mathcal{O}_{y,Y}$-module. For if $g_y \in \mathcal{O}_{y,Y}$ we have a representative $g \in I(U)$ for some open neighborhood $U$ of $y$. Then $g \circ \psi \in I(\psi^{-1}(U))$. If $\xi_y \in \psi_{(l)}(S)_{(y)}$ and $U$ is sufficiently small we can find a representative $\xi \in H^l(\psi^{-1}(U), S)$ for $\xi_y$. Then we put $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$. Now we form $\psi_{(l)}(S) = \bigcup_{y \in Y} \psi_{(l)}(S)_{(y)}$ where we introduce a sheaf topology. A base of the open sets are $\{ \xi_y : y \in U \}$ for $\xi \in H^l(\psi^{-1}(U), S)$. If $\xi \in H^l(X, S)$ then the map $y \to \xi_y$ is a cross-section in $\psi_{(l)}(S)$. We call it the direct image of $\xi$ and denote it by $\psi_{(l)}(\xi)$. The sheaf $\psi_{(l)}(S)$ is the direct image sheaf of $S$ of dimension $l$. Our main problem is to decide whether $\psi_{(l)}(S)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules if $S$ is a coherent analytic sheaf on $X$.

## A very special case

We shall consider a special case where our main problem is easily solved. Let $X_0$ be a compact analytic manifold of pure dimension $m - n$. We put $E^n(\rho_0) = \{ (t_1 \dots t_n) \in C^n ;\ |t_i| < \rho_i^0 \}$. Here $\rho_0 = (\rho_1^0 \dots \rho_n^0)$ is a fixed $n$-tuple of strictly positive numbers. Let $X = E^n(\rho_0) \times X_0$ and $X(\rho) = E^n(\rho) \times X_0$ for $\rho \leqslant \rho_0$. We see that $X$ is an analytic manifold of pure dimension $m$. Let $\psi : X \to E^n(\rho_0)$ be the projection map. Now $X$ is fibered by the fibers $\psi^{-1}(t) = X(t) = \{ t \} \times X_0 \cong X_0$ for $t \in E^n(\rho_0)$. We take the sheaf $S$ to be $S = (q\mathcal{O})_X$. With these notations we can state the following.

*Theorem:* The direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is a coherent sheaf of $\mathcal{O}_{E^n(\rho_0)}$-modules for every $l \geqslant 0$.

*Proof.* Because $X_0$ is a compact analytic manifold we can find a finite Stein covering $\mathfrak{U} = \{ U_1 \dots U_{\iota^*} \}$ of $X_0$. Let us put $\hat{U}_\iota = E^n(\rho_0) \times U_\iota$, then we see that $\hat{\mathfrak{U}} = \{ \hat{U}_1 \dots \hat{U}_{\iota^*} \}$ is a Stein covering of $X$. Let $\hat{\xi} = \{ \hat{\xi}_{\iota_0 \dots \iota_l} \} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Now $\hat{\xi}_{\iota_0 \dots \iota_l}$ is a $q$-tuple of holomorphic functions on $E^n(\rho_0) \times U_{\iota_0 \dots \iota_l}$. Hence $\hat{\xi}_{\iota_0 \dots \iota_l}$ admits a Taylor series of the form $\hat{\xi}_{\iota_0 \dots \iota_l} = \sum_{|\nu|=0}^{\infty} \xi^{(\nu)}_{\iota_0 \dots \iota_l} (t/\rho_0)^\nu$ where $\nu = (\nu_1, \dots, \nu_n)$, $|\nu| = \nu_1 + \dots + \nu_n$ and $(t/\rho)^\nu = (t_1/\rho_1)^{\nu_1} \dots (t_n/\rho_n)^{\nu_n}$. The uniqueness of a Taylor series shows that $\{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \}$ is an alternating cochain over $\mathfrak{U}$. Putting $\xi_{(\nu)} = \{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \} \in C^l(\mathfrak{U}, (q\mathcal{O})_X)$ we may write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$. Introducing the map $(\nu) : \hat{\xi} \to \xi_{(\nu)}$ we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{O})_X) \\ (\nu)\downarrow & & \downarrow(\nu) \\ C^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{O})_{X_0}) . \end{array}$$

We now need a *theorem of Cartan-Serre:* Let $X_0$ be a compact analytic manifold. Then, for any coherent analytic sheaf $S$ the set $H^p(X_0, S)$ is a finite dimensional vector space for all $p \geqslant 0$.