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spread holomorphically, i.e. for any  $x \in X$  there exists  $f_1 \dots f_N \in I(X)$  such that  $x$  is an isolated common zero of  $f_1 \dots f_N$ .

Let  $X$  be a complex analytic manifold. A Stein covering  $\mathfrak{U} = \{U_i\}_{i \in J}$  of  $X$  is an open covering of  $X$  such that every  $U_i$  is Stein. We shall often use the following result:

*Leray's Theorem:* If  $\mathfrak{U}$  is a Stein covering of  $X$  then  $H^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$  is an isomorphism for every coherent analytic sheaf  $S$ .

The isomorphism between  $H^l(\mathfrak{U}, S)$  and  $H^l(X, S)$  means the following: If  $\underline{\xi} \in H^l(X, S)$  there exists  $\xi \in Z^l(\mathfrak{U}, S)$  such that  $\xi$  maps into  $\underline{\xi}$  under the natural homomorphism  $Z^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$  and moreover if  $\underline{\xi} \in Z^l(\mathfrak{U}, S)$  is mapped into zero in  $H^l(X, S)$  there exist  $\eta \in C^{l-1}(\mathfrak{U}, S)$  such that  $\xi = \delta\eta$  in  $C^l(\mathfrak{U}, S)$ .

#### DIRECT IMAGES OF SHEAVES

Let  $X$  and  $Y$  be complex analytic manifolds. Let  $\psi : X \rightarrow Y$  be a holomorphic map and let  $S$  be an analytic sheaf on  $X$ . Now  $X$  is fibered by the fibers  $X(y) = \psi^{-1}(y)$  for  $y \in Y$ . Let  $U$  be an open neighborhood of a point  $y \in Y$ , then  $V = \psi^{-1}(U)$  is an open set in  $X$ . Hence  $V$  is a complex analytic manifold and the restriction of  $S$  to  $V$  gives an analytic sheaf on  $V$ . We can now define  $H^l(V, S)$ . Let us put  $H_y^l = \bigcup_U H^l(\psi^{-1}(U), S)$

where  $U$  runs over all open neighborhoods of  $y$  in  $Y$ . In  $H_y^l$  we introduce an equivalence relation as follows:  $\xi_1 \in H^l(\psi^{-1}(U_1), S)$  and  $\xi_2 \in H^l(\psi^{-1}(U_2), S)$  are equivalent iff there exists  $U = U(y)$  in  $Y$  such that  $U \subset U_1 \cap U_2$  and  $\xi_1|_{\psi^{-1}(U)} = \xi_2|_{\psi^{-1}(U)}$  in  $H^l(\psi^{-1}(U), S)$ . We let  $\psi_{(l)}(S)_{(y)}$  denote the set of equivalence classes in  $H_y^l$ . The equivalence class generated by  $\xi \in H^l(\psi^{-1}(U), S)$  is denoted by  $\xi_y$ . The set  $\psi_{(l)}(S)_{(y)}$  is called the set of germs of cohomology classes of dimension  $l$  along the fiber  $X(y)$ . Now  $\psi_{(l)}(S)_{(y)}$  is an  $\mathcal{O}_{y,Y}$ -module. For if  $g_y \in \mathcal{O}_{y,Y}$  we have a representative  $g \in I(U)$  for some open neighborhood  $U$  of  $y$ . Then  $g \circ \psi \in I(\psi^{-1}(U))$ . If  $\xi_y \in \psi_{(l)}(S)_{(y)}$  and  $U$  is sufficiently small we can find a representative  $\xi \in H^l(\psi^{-1}(U), S)$  for  $\xi_y$ . Then we put  $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$ . Now we form  $\psi_{(l)}(S) = \bigcup_{y \in Y} \psi_{(l)}(S)_{(y)}$  where we introduce a sheaf topology.

A base of the open sets are  $\{\xi_y : y \in U\}$  for  $\xi \in H^l(\psi^{-1}(U), S)$ . If  $\xi \in H^l(X, S)$  then the map  $y \rightarrow \xi_y$  is a cross-section in  $\psi_{(l)}(S)$ . We call it the direct image of  $\xi$  and denote it by  $\psi_{(l)}(\xi)$ . The sheaf  $\psi_{(l)}(S)$  is the direct

image sheaf of  $S$  of dimension  $l$ . Our main problem is to decide whether  $\psi_{(l)}(S)$  is a coherent analytic sheaf of  $\mathcal{O}_Y$ -modules if  $S$  is a coherent analytic sheaf on  $X$ .

A VERY SPECIAL CASE

We shall consider a special case where our main problem is easily solved. Let  $X_0$  be a compact analytic manifold of pure dimension  $m - n$ . We put  $E^n(\rho_0) = \{ (t_1 \dots t_n) \in \mathbf{C}^n ; |t_i| < \rho_i^0 \}$ . Here  $\rho_0 = (\rho_1^0 \dots \rho_n^0)$  is a fixed  $n$ -tuple of strictly positive numbers. Let  $X = E^n(\rho_0) \times X_0$  and  $X(\rho) = E^n(\rho) \times X_0$  for  $\rho \leq \rho_0$ . We see that  $X$  is an analytic manifold of pure dimension  $m$ . Let  $\psi : X \rightarrow E^n(\rho_0)$  be the projection map. Now  $X$  is fibered by the fibers  $\psi^{-1}(t) = X(t) = \{t\} \times X_0 \cong X_0$  for  $t \in E^n(\rho_0)$ . We take the sheaf  $S$  to be  $S = (q\mathcal{C})_X$ . With these notations we can state the following.

*Theorem:* The direct image sheaf  $\psi_{(l)}((q\mathcal{C})_X)$  is a coherent sheaf of  $\mathcal{O}_{E^n(\rho_0)}$ -modules for every  $l \geq 0$ .

*Proof.* Because  $X_0$  is a compact analytic manifold we can find a finite Stein covering  $\mathfrak{U} = \{U_1 \dots U_{l_*}\}$  of  $X_0$ . Let us put  $\hat{U}_i = E^n(\rho_0) \times U_i$ , then we see that  $\hat{\mathfrak{U}} = \{\hat{U}_1 \dots \hat{U}_{l_*}\}$  is a Stein covering of  $X$ . Let  $\hat{\xi} = \{\hat{\xi}_{i_0 \dots i_l}\} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{C})_X)$ . Now  $\hat{\xi}_{i_0 \dots i_l}$  is a  $q$ -tuple of holomorphic functions on  $E^n(\rho_0) \times U_{i_0 \dots i_l}$ . Hence  $\hat{\xi}_{i_0 \dots i_l}$  admits a Taylor series of the form  $\hat{\xi}_{i_0 \dots i_l} = \sum_{|v|=0}^{\infty} \xi_{i_0 \dots i_l}^{(v)} (t/\rho_0)^v$  where  $v = (v_1, \dots, v_n)$ ,  $|v| = v_1 + \dots + v_n$  and  $(t/\rho)^v = (t_1/\rho_1)^{v_1} \dots (t_n/\rho_n)^{v_n}$ . The uniqueness of a Taylor series shows that  $\{\xi_{i_0 \dots i_l}^{(v)}\}$  is an alternating cochain over  $\mathfrak{U}$ . Putting  $\xi_{(v)} = \{\xi_{i_0 \dots i_l}^{(v)}\} \in C^l(\mathfrak{U}, (q\mathcal{C})_X)$  we may write  $\hat{\xi} = \sum \xi_{(v)} (t/\rho)^v$ . Introducing the map  $(v) : \hat{\xi} \rightarrow \xi_{(v)}$  we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{C})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{C})_X) \\ (v)\downarrow & & \downarrow (v) \\ C^l(\mathfrak{U}, (q\mathcal{C})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{C})_{X_0}). \end{array}$$

We now need a *theorem of Cartan-Serre*: Let  $X_0$  be a compact analytic manifold. Then, for any coherent analytic sheaf  $S$  the set  $H^p(X_0, S)$  is a finite dimensional vector space for all  $p \geq 0$ .