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The previous lemma shows that $\overset{*}{\xi}_{(\nu)} = \sum a_{\nu\lambda}\, \mathfrak{b}_\lambda + \delta\eta_\nu$ where $\eta_\nu \in C^{l-1}(\mathfrak{B})$ with $\| \eta_\nu \mid \mathfrak{B}_1 \| \leqslant K \| \overset{*}{\xi}_{(\nu)} \|$ and $| a_{\nu\lambda} | \leqslant K \| \overset{*}{\xi}_{(\nu)} \|$. Let us put $a_\lambda = \sum a_{\nu\lambda}\, (t/\rho_2)^\nu$ and $\hat{\eta} = \sum \eta_\nu\, (t/\rho_2)^\nu$. We see that $\hat{\eta} \in C^{l-1}(\mathfrak{B}_1(\rho_2))$ and $\hat{a}_\lambda \in I(E^n(\rho_2))$. An easy computation gives $\hat{\xi}_1 \mid \mathfrak{B}_1(\rho_2) = \sum \hat{a}_\lambda\, \hat{\mathfrak{b}}_\lambda \mid \mathfrak{B}_1(\rho_2) + \delta\, \hat{\eta}$. It follows by definition that $\xi_0 = \sum a_\lambda\, \hat{\mathbf{b}}_\lambda$. We have now proved that $\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ at the origin. It follows in the same way that $\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ for every $t \in E^n(\rho_0)$ because it is enough to do everything in a polydisc around $t$. Now we also prove that the sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is free, i.e. there are no relations between $\hat{\mathbf{b}}_1 \ldots \hat{\mathbf{b}}_r$ at any point. Say for example that $a_1\, \hat{\mathbf{b}}_1 + \ldots + a_r\, \hat{\mathbf{b}}_r = 0$ at $\psi_{(l)}((q\mathcal{O})_X)_{(0)}$ where $a_\iota$ are germs of analytic functions at the origin in $E^n(\rho_0)$. Hence $\tilde{a}_1\, \hat{\mathfrak{b}}_1 + \ldots + \tilde{a}_r\, \hat{\mathfrak{b}}_r = 0$ in $H^l(X(\rho), (q\mathcal{O})_X)$ for some $\rho > 0$ with $\tilde{a}_\iota \in I(E^n(\rho))$. It follows that $\sum \tilde{a}_\nu\, \hat{\mathfrak{b}}_\nu = \delta\, \hat{\xi}$ in $X(\rho)$ for some $\hat{\xi} \in C^{l-1}(\hat{\mathfrak{U}}(\rho), (q\mathcal{O})_X)$. Take a point $t \in E^n(\rho)$ where some $\tilde{a}_\nu \neq 0$. Now we see that on $\{ t \} \times X_0$ we have $\tilde{a}_1(t)\, \mathfrak{b}_1 + \ldots + \tilde{a}_r(t)\, \mathfrak{b}_r = \partial\, \hat{\xi} \mid \{ t \} \times X_0 \in C^{l-1}(\mathfrak{U}, (q\mathcal{O})_{X_0})$. This gives a contradiction to the fact that $\mathbf{b}_1 \ldots \mathbf{b}_r$ are a base of $H^l(X_0, (q\mathcal{O})_{X_0})$.

## Measure charts

Let $X$ be a connected complex analytic manifold of dimension $m$. Let $F$ be a holomorphic vector bundle of rank $q$ on $X$ and $\mathbf{F}$ the sheaf of holomorphic crossections in $F$. This sheaf is locally free. A regular proper holomorphic map $\psi\colon X \to E^n$ is given. Let us put $X_0 = \psi^{-1}(0)$. Now $X_0$ is a compact analytic manifold of dimension $m - n$. We now introduce special open coverings around $X_0$ in $X$.

*Definition.* A measure chart $\mathscr{W} = (\hat{W}, \Phi, \Theta, \rho)$ is a quadruple satisfying the conditions:

1) $\hat{W} \subset X$ is open and $W = \hat{W} \cap X_0$ is Stein.

2) $\Phi\colon \hat{W} \to E^n(\rho) \times W$ is a biholomorphic map such that the following diagram is commutative:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{\Phi} & E^n(\rho) \times W \\ & \psi \searrow \quad \swarrow \pi & \\ & E^n(\rho)\,. & \end{array}$$

Here $\pi$ is the projection map.

3) $\Theta: F|\hat{W} \to \hat{W} \times \mathbf{C}^q$ is a trivialization of $F$ on $\hat{W}$.

If $\mathscr{W}$ is a given measure chart on $X$ we can identify the sheaf $(\hat{W}, \mathbf{F}|\hat{W})$ of $\mathcal{O}_X$-modules with the sheaf $(W \times E^n(\rho), q\,\mathcal{O})$ using $\Phi$ and $\Theta$. If $U \subset W$ is open and $\rho' \leqslant \rho$ we put $\hat{U}(\rho') = \Phi^{-1}(U \times E^n(\rho'))$. Hence if $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we can identify $\hat{s}$ with an element of $\Gamma(U \times E^n(\rho'), q\,\mathcal{O})$. We shall simply denote this element of $\Gamma(U \times E^n(\rho'), q\,\mathcal{O})$ by the same letter $\hat{s}$. Now we can expand $\hat{s}$ in a Taylor series: $\hat{s} = \sum_{|\nu|=0}^{\infty} s_\nu (t/\rho')^\nu$ where $s_\nu \in qI(U)$.

*Definition of a norm.* When $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we put $\|\hat{s}\| = = \sup_\nu |s_\nu(U)|$.

Strictly speaking the norm $\|\hat{s}\|$ is taken with respect to the measure chart $\mathscr{W}$.

It is not hard to see that for every point $x \in X_0$ there exists a measure chart $\mathscr{W}$ such that $x \in \hat{W}$. In particular we can cover $X_0$ by finitely many measure charts $\mathscr{W}_\iota = (\hat{W}_\iota, \Phi_\iota, \Theta_\iota, \rho_\iota)$, i.e. $X_0 \subset\subset \bigcup_1^{\iota^*} \hat{W}_\iota$. We remark that it follows that $X(\rho) = \psi^{-1}(E^n(\rho)) \subset\subset \bigcup_1^{\iota^*} \hat{W}_\iota$ for some $\rho > 0$ with $\rho \leqslant \rho_\iota$ because $\psi$ is a proper map. The collection $\mathscr{W} = \{\mathscr{W}_\iota\}_1^{\iota^*}$ is called an atlas around $X_0$. From now on $\mathscr{W}$ is a fixed atlas.

*Measure coverings.* We shall define measure coverings with respect to the given atlas $\mathscr{W}$ above. If $U \subset W_\iota$ is open we put $(U)_\iota(\rho) = = \Phi_\iota^{-1}(U \times E^n(\rho))$ when $\rho \leqslant \rho_\iota$. We see that $(U)_\iota(\rho) \subset \hat{W}_\iota$ and $(U)_\iota(\rho)$ is Stein if $U$ is Stein. Let $\mathfrak{U} = \{U_\iota\}_1^{\iota^*}$ be a Stein covering of $X_0$ with $U_\iota \subset\subset W_\iota$ for each $\iota$. Let $\rho > 0$ with $\rho < \min_\iota \rho_\iota$. We put $\hat{U}_\iota(\rho) = (U_\iota)_\iota(\rho)$. We see that $\hat{U}_\iota(\rho) \subset\subset \hat{W}_\iota$ and $\hat{U}_\iota(\rho)$ are Stein. It is now required that $\hat{\mathfrak{U}}(\rho) =$

$= \{ \hat{U}_\iota(\rho) \}_1^{\iota^*}$ is a Stein covering of $X(\rho)$. We say then that $\hat{\mathfrak{U}}(\rho)$ is a measure covering of $X(\rho)$.

*Admissible refinements of measure coverings.* Let $\hat{\mathfrak{U}}(\rho)$ and $\hat{\mathfrak{U}}^*(\rho)$ be two measure coverings of $X(\rho)$. We say that $\hat{\mathfrak{U}}^*(\rho)$ is an admissible refinement of $\hat{\mathfrak{U}}(\rho)$ if the following conditions hold:

1) $U^*_\iota \subset\subset U_\iota$ for each $\iota$.

2) If $U^*_{\iota_0 \dots \iota_\lambda} = U^*_{\iota_0} \cap \dots \cap U^*_{\iota_\lambda}$ we put $(U^*_{\iota_0 \dots \iota_\lambda})_\nu = \Phi_\nu^{-1}(U^*_{\iota_0 \dots \iota_\lambda} \times E^n(\rho))$ for each $\nu \in \{ \iota_0 \dots \iota_\lambda \}$. It is now required that $(U^*_{\iota_0 \dots \iota_\lambda})_\nu \subset (U_{\iota_0 \dots \iota_\lambda})_\mu$ for all $\nu, \mu \in \{ \iota_0 \dots \iota_\lambda \}$.

3) $\hat{U}^*_{\iota_0 \dots \iota_\lambda} = \hat{U}^*_{\iota_0} \cap \dots \cap \hat{U}^*_{\iota_\lambda} \subset (U_{\iota_0 \dots \iota_\lambda})_\mu$ for each $\mu \in \{ \iota_0 \dots \iota_\lambda \}$.

## Existence of admissible refinements of measure coverings

*Existence Theorem.* For every fixed integer $s$ we can find, for some $\rho > 0$, a sequence $\mathfrak{U}_s \ll \mathfrak{U}_{s-1} \ll \dots \ll \mathfrak{U}_1 \ll \mathfrak{U}_0$ of finer measure coverings of $X(\rho)$ each of which is an admissible refinement of the following.

*Proof.* We first construct a measure covering of $X(\rho)$ for some $\rho < \min \rho_\iota$. Let $\mathfrak{U}_0 = \{ \mathfrak{U}_\iota \}_1^{\iota^*}$ be a Stein covering of $X_0$ such that $U_\iota \subset\subset W_\iota$ for $\iota \in \{ 1, \dots, \iota^* \}$. Choose a fixed $\rho_0 < \min \rho_\iota$. Now the open sets $\Phi_\iota^{-1}(U_\iota \times E^n(\rho_0))$ cover $X_0$ and hence they also cover $X(\rho)$ for some sufficiently small $\rho$. Hence $\mathfrak{U}_0$ defines a measure covering of $X(\rho)$. It is also clear that $\mathfrak{U}_0$ defines a measure covering of $X(\rho')$ for each $\rho' \leqslant \rho$. Let us now construct $\mathfrak{U}_1$. We let $\mathfrak{U}^* = \{ U^*_\iota \}_1^{\iota^*}$ be a Stein covering such that $U^*_\iota \subset\subset U_\iota$ always holds. Now we can find $\rho_1 \leqslant \rho$ such that $\{ \hat{U}^*_\iota(\rho_1) = \Phi_\iota^{-1}(U^*_\iota \times E^n(\rho_1)) \}_1^{\iota^*}$ cover $X(\rho_1)$. Hence $\hat{\mathfrak{U}}^*(\rho_1)$ and $\hat{\mathfrak{U}}(\rho_1)$ are measure coverings of $X(\rho_1)$. But we do not yet know if $\hat{\mathfrak{U}}^*(\rho_1) \ll \hat{\mathfrak{U}}(\rho_1)$. We claim that if $\rho_2 \leqslant \rho_1$ is sufficiently small then $\hat{\mathfrak{U}}^*(\rho_2) \ll \hat{\mathfrak{U}}(\rho_2)$. For suppose this is false. Say that 2) fails for $\hat{\mathfrak{U}}^*(\rho_2)$ and $\hat{\mathfrak{U}}(\rho_2)$ when $0 < \rho_2 \leqslant \rho_1$. Hence $\Phi_\nu^{-1}(U^*_{\iota_0 \dots \iota_\lambda} \times E^n(\rho_2)) - \Phi_\mu^{-1}(U_{\iota_0 \dots \iota_\lambda} \times E^n(\rho_2))$ are non empty for suitable indices while $\rho_2 \to 0$. Choose a point $x_t$ from each of these sets. Because $x_t \in X(\rho_1)$ which is relatively compact we may assume that $x_t \to x_0$. Obviously we get $x_0 \in \overline{U^*_{\iota_0 \dots \iota_\lambda}} - U_{\iota_0 \dots \iota_\lambda}$, a contradic-