

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COHERENCE OF DIRECT IMAGES
Autor: Grauert, H.
Kapitel: Measure charts
DOI: <https://doi.org/10.5169/seals-42345>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 28.01.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The previous lemma shows that $\xi_{(v)}^* = \sum a_{v\lambda} \mathfrak{b}_\lambda + \delta \eta_v$ where $\eta_v \in C^{l-1}(\mathfrak{B})$ with $\|\eta_v|_{\mathfrak{B}_1}\| \leq K \|\xi_{(v)}^*\|$ and $|a_{v\lambda}| \leq K \|\xi_{(v)}^*\|$. Let us put $a_\lambda = \sum a_{v\lambda} (t/\rho_2)^v$ and $\hat{\eta} = \sum \eta_v (t/\rho_2)^v$. We see that $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}_1(\rho_2))$ and $a_\lambda \in I(E^n(\rho_2))$. An easy computation gives $\hat{\xi}_1|_{\hat{\mathfrak{B}}_1(\rho_2)} = \sum a_\lambda \hat{\mathfrak{b}}_\lambda|_{\hat{\mathfrak{B}}_1(\rho_2)} + \delta \hat{\eta}$. It follows by definition that $\xi_0 = \sum a_\lambda \hat{\mathfrak{b}}_\lambda$. We have now proved that $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ generate $\psi_{(t)}((q\mathcal{O})_X)$ at the origin. It follows in the same way that $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ generate $\psi_{(t)}((q\mathcal{O})_X)$ for every $t \in E^n(\rho_0)$ because it is enough to do everything in a polydisc around t . Now we also prove that the sheaf $\psi_{(t)}((q\mathcal{O})_X)$ is free, i.e. there are no relations between $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ at any point. Say for example that $a_1 \hat{\mathfrak{b}}_1 + \dots + a_r \hat{\mathfrak{b}}_r = 0$ at $\psi_{(t)}((q\mathcal{O})_X)_{(0)}$ where a_i are germs of analytic functions at the origin in $E^n(\rho_0)$. Hence $\tilde{a}_1 \hat{\mathfrak{b}}_1 + \dots + \tilde{a}_r \hat{\mathfrak{b}}_r = 0$ in $H^l(X(\rho), (q\mathcal{O})_X)$ for some $\rho > 0$ with $\tilde{a}_i \in I(E^n(\rho))$. It follows that $\sum \tilde{a}_v \hat{\mathfrak{b}}_v = \delta \hat{\xi}$ in $X(\rho)$ for some $\hat{\xi} \in C^{l-1}(\hat{\mathfrak{U}}(\rho), (q\mathcal{O})_X)$. Take a point $t \in E^n(\rho)$ where some $\tilde{a}_v \neq 0$. Now we see that on $\{t\} \times X_0$ we have $\tilde{a}_1(t) \mathfrak{b}_1 + \dots + \tilde{a}_r(t) \mathfrak{b}_r = \partial \hat{\xi}|_{\{t\} \times X_0} \in C^{l-1}(\mathfrak{U}, (q\mathcal{O})_{X_0})$. This gives a contradiction to the fact that $\mathfrak{b}_1 \dots \mathfrak{b}_r$ are a base of $H^l(X_0, (q\mathcal{O})_{X_0})$.

MEASURE CHARTS

Let X be a connected complex analytic manifold of dimension m . Let F be a holomorphic vector bundle of rank q on X and \mathbf{F} the sheaf of holomorphic crosssections in F . This sheaf is locally free. A regular proper holomorphic map $\psi: X \rightarrow E^n$ is given. Let us put $X_0 = \psi^{-1}(0)$. Now X_0 is a compact analytic manifold of dimension $m - n$. We now introduce special open coverings around X_0 in X .

Definition. A measure chart $\mathcal{W} = (\hat{W}, \Phi, \Theta, \rho)$ is a quadruple satisfying the conditions:

- 1) $\hat{W} \subset X$ is open and $W = \hat{W} \cap X_0$ is Stein.
- 2) $\Phi: \hat{W} \rightarrow E^n(\rho) \times W$ is a biholomorphic map such that the following diagram is commutative:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{\Phi} & E^n(\rho) \times W \\ \psi \searrow & & \swarrow \pi \\ & & E^n(\rho). \end{array}$$

Here π is the projection map.

3) $\Theta: F|_{\hat{W}} \rightarrow \hat{W} \times \mathbf{C}^q$ is a trivialization of F on \hat{W} .

If \mathcal{W} is a given measure chart on X we can identify the sheaf $(\hat{W}, F|_{\hat{W}})$ of \mathcal{C}_X -modules with the sheaf $(W \times E^n(\rho), q\mathcal{O})$ using Φ and Θ . If $U \subset W$ is open and $\rho' \leq \rho$ we put $\hat{U}(\rho') = \Phi^{-1}(U \times E^n(\rho'))$. Hence if $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we can identify \hat{s} with an element of $\Gamma(U \times E^n(\rho'), q\mathcal{O})$. We shall simply denote this element of $\Gamma(U \times E^n(\rho'), q\mathcal{O})$ by the same letter \hat{s} . Now we can expand \hat{s} in a Taylor series: $\hat{s} = \sum_{|v|=0}^{\infty} s_v (t/\rho')^v$ where $s_v \in qI(U)$.

Definition of a norm. When $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we put $\|\hat{s}\| = \sup_v |s_v(U)|$.

Strictly speaking the norm $\|\hat{s}\|$ is taken with respect to the measure chart \mathcal{W} .

It is not hard to see that for every point $x \in X_0$ there exists a measure chart \mathcal{W} such that $x \in \hat{W}$. In particular we can cover X_0 by finitely many measure charts $\mathcal{W}_\iota = (\hat{W}_\iota, \Phi_\iota, \Theta_\iota, \rho_\iota)$, i.e. $X_0 \subset \subset \bigcup_1^{\iota^*} \hat{W}_\iota$. We remark that it

follows that $X(\rho) = \psi^{-1}(E^n(\rho)) \subset \subset \bigcup_1^{\iota^*} \hat{W}_\iota$ for some $\rho > 0$ with $\rho \leq \rho_\iota$ because ψ is a proper map. The collection $\mathcal{W} = \{\mathcal{W}_\iota\}_1^{\iota^*}$ is called an atlas around X_0 . From now on \mathcal{W} is a fixed atlas.

Measure coverings. We shall define measure coverings with respect to the given atlas \mathcal{W} above. If $U \subset W_\iota$ is open we put $(U)_\iota(\rho) = \Phi_\iota^{-1}(U \times E^n(\rho))$ when $\rho \leq \rho_\iota$. We see that $(U)_\iota(\rho) \subset \hat{W}_\iota$ and $(U)_\iota(\rho)$ is Stein if U is Stein. Let $\mathfrak{U} = \{U_\iota\}_1^{\iota^*}$ be a Stein covering of X_0 with $U_\iota \subset \subset W_\iota$ for each ι . Let $\rho > 0$ with $\rho < \min_\iota \rho_\iota$. We put $\hat{U}_\iota(\rho) = (U_\iota)_\iota(\rho)$. We see that $\hat{U}_\iota(\rho) \subset \subset \hat{W}_\iota$ and $\hat{U}_\iota(\rho)$ are Stein. It is now required that $\hat{\mathfrak{U}}(\rho) =$

$= \{ \hat{U}_i(\rho) \}_1^{i^*}$ is a Stein covering of $X(\rho)$. We say then that $\hat{\mathfrak{U}}(\rho)$ is a measure covering of $X(\rho)$.

Admissible refinements of measure coverings. Let $\hat{\mathfrak{U}}(\rho)$ and $\hat{\mathfrak{U}}^*(\rho)$ be two measure coverings of $X(\rho)$. We say that $\hat{\mathfrak{U}}^*(\rho)$ is an admissible refinement of $\hat{\mathfrak{U}}(\rho)$ if the following conditions hold:

- 1) $U_i^* \subset \subset U_i$ for each i .
- 2) If $U_{i_0 \dots i_\lambda}^* = U_{i_0}^* \cap \dots \cap U_{i_\lambda}^*$ we put $(U_{i_0 \dots i_\lambda}^*)_v = \Phi_v^{-1}(U_{i_0 \dots i_\lambda}^* \times E^n(\rho))$ for each $v \in \{i_0 \dots i_\lambda\}$. It is now required that $(U_{i_0 \dots i_\lambda}^*)_v \subset (U_{i_0 \dots i_\lambda}^*)_\mu$ for all $v, \mu \in \{i_0 \dots i_\lambda\}$.
- 3) $\hat{U}_{i_0 \dots i_\lambda}^* = \hat{U}_{i_0}^* \cap \dots \cap \hat{U}_{i_\lambda}^* \subset (U_{i_0 \dots i_\lambda}^*)_\mu$ for each $\mu \in \{i_0 \dots i_\lambda\}$.

EXISTENCE OF ADMISSIBLE REFINEMENTS OF MEASURE COVERINGS

Existence Theorem. For every fixed integer s we can find, for some $\rho > 0$, a sequence $\mathfrak{U}_s \ll \mathfrak{U}_{s-1} \ll \dots \ll \mathfrak{U}_1 \ll \mathfrak{U}_0$ of finer measure coverings of $X(\rho)$ each of which is an admissible refinement of the following.

Proof. We first construct a measure covering of $X(\rho)$ for some $\rho < \min \rho_i$. Let $\mathfrak{U}_0 = \{ \mathfrak{U}_i \}_1^{i^*}$ be a Stein covering of X_0 such that $U_i \subset \subset W_i$ for $i \in \{1, \dots, i^*\}$. Choose a fixed $\rho_0 < \min \rho_i$. Now the open sets $\Phi_i^{-1}(U_i \times E^n(\rho_0))$ cover X_0 and hence they also cover $X(\rho)$ for some sufficiently small ρ . Hence \mathfrak{U}_0 defines a measure covering of $X(\rho)$. It is also clear that \mathfrak{U}_0 defines a measure covering of $X(\rho')$ for each $\rho' \leq \rho$. Let us now construct \mathfrak{U}_1 . We let $\mathfrak{U}^* = \{ U_i^* \}_1^{i^*}$ be a Stein covering such that $U_i^* \subset \subset U_i$ always holds. Now we can find $\rho_1 \leq \rho$ such that $\{ \hat{U}_i^*(\rho_1) = \Phi_i^{-1}(U_i^* \times E^n(\rho_1)) \}_1^{i^*}$ cover $X(\rho_1)$. Hence $\hat{\mathfrak{U}}^*(\rho_1)$ and $\hat{\mathfrak{U}}(\rho_1)$ are measure coverings of $X(\rho_1)$. But we do not yet know if $\hat{\mathfrak{U}}^*(\rho_1) \ll \hat{\mathfrak{U}}(\rho_1)$. We claim that if $\rho_2 \leq \rho_1$ is sufficiently small then $\hat{\mathfrak{U}}^*(\rho_2) \ll \hat{\mathfrak{U}}(\rho_2)$. For suppose this is false. Say that 2) fails for $\hat{\mathfrak{U}}^*(\rho_2)$ and $\hat{\mathfrak{U}}(\rho_2)$ when $0 < \rho_2 \leq \rho_1$. Hence $\Phi_v^{-1}(U_{i_0 \dots i_\lambda}^* \times E^n(\rho_2)) - \Phi_\mu^{-1}(U_{i_0 \dots i_\lambda} \times E^n(\rho_2))$ are non empty for suitable indices while $\rho_2 \rightarrow 0$. Choose a point x_t from each of these sets. Because $x_t \in X(\rho_1)$ which is relatively compact we may assume that $x_t \rightarrow x_0$. Obviously we get $x_0 \in \overline{U_{i_0 \dots i_\lambda}^*} - U_{i_0 \dots i_\lambda}$, a contradic-