Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COHERENCE OF DIRECT IMAGES

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Kapitel: Existence of admissible refinements of measure coverings

DOI: https://doi.org/10.5169/seals-42345

$= \{ \hat{U}_\iota(\rho) \}_1^{\iota^*}$ is a Stein covering of $X(\rho)$. We say then that $\hat{\mathfrak{U}}(\rho)$ is a measure covering of $X(\rho)$.

Admissible refinements of measure coverings. Let $\hat{\mathfrak{U}}(\rho)$ and $\hat{\mathfrak{U}}^*(\rho)$ be two measure coverings of $X(\rho)$. We say that $\hat{\mathfrak{U}}^*(\rho)$ is an admissible refinement of $\hat{\mathfrak{U}}(\rho)$ if the following conditions hold:

1) $U^*_\iota \subset\subset U_\iota$ for each ι.

2) If $U^*_{\iota_0 \ldots \iota_\lambda} = U^*_{\iota_0} \cap \ldots \cap U^*_{\iota_\lambda}$ we put $(U^*_{\iota_0 \ldots \iota_\lambda})_\nu = \Phi_\nu^{-1}(U^*_{\iota_0 \ldots \iota_\lambda} \times E^n(\rho))$ for each $\nu \in \{ \iota_0 \ldots \iota_\lambda \}$. It is now required that $(U^*_{\iota_0 \ldots \iota_\lambda})_\nu \subset (U_{\iota_0 \ldots \iota_\lambda})_\mu$ for all $\nu, \mu \in \{ \iota_0 \ldots \iota_\lambda \}$.

3) $\hat{U}^*_{\iota_0 \ldots \iota_\lambda} = \hat{U}^*_{\iota_0} \cap \ldots \cap \hat{U}^*_{\iota_\lambda} \subset (U_{\iota_0 \ldots \iota_\lambda})_\mu$ for each $\mu \in \{ \iota_0 \ldots \iota_\lambda \}$.

Existence of admissible refinements of measure coverings

Existence Theorem. For every fixed integer s we can find, for some $\rho > 0$, a sequence $\mathfrak{U}_s \ll \mathfrak{U}_{s-1} \ll \ldots \ll \mathfrak{U}_1 \ll \mathfrak{U}_0$ of finer measure coverings of $X(\rho)$ each of which is an admissible refinement of the following.

Proof. We first construct a measure covering of $X(\rho)$ for some $\rho < \min \rho_\iota$. Let $\mathfrak{U}_0 = \{ \mathfrak{U}_\iota \}_1^{\iota^*}$ be a Stein covering of X_0 such that $U_\iota \subset\subset W_\iota$ for $\iota \in \{ 1, \ldots, \iota^* \}$. Choose a fixed $\rho_0 < \min \rho_\iota$. Now the open sets $\Phi_\iota^{-1}(U_\iota \times E^n(\rho_0))$ cover X_0 and hence they also cover $X(\rho)$ for some sufficiently small ρ. Hence $\mathfrak{U}_0$ defines a measure covering of $X(\rho)$. It is also clear that $\mathfrak{U}_0$ defines a measure covering of $X(\rho')$ for each $\rho' \leqslant \rho$. Let us now construct $\mathfrak{U}_1$. We let $\mathfrak{U}^* = \{ U^*_\iota \}_1^{\iota^*}$ be a Stein covering such that $U^*_\iota \subset\subset U_\iota$ always holds. Now we can find $\rho_1 \leqslant \rho$ such that $\{ \hat{U}^*_\iota(\rho_1) = \Phi_\iota^{-1}(U^*_\iota \times E^n(\rho_1)) \}_1^{\iota^*}$ cover $X(\rho_1)$. Hence $\hat{\mathfrak{U}}^*(\rho_1)$ and $\hat{\mathfrak{U}}(\rho_1)$ are measure coverings of $X(\rho_1)$. But we do not yet know if $\hat{\mathfrak{U}}^*(\rho_1) \ll \hat{\mathfrak{U}}(\rho_1)$. We claim that if $\rho_2 \leqslant \rho_1$ is sufficiently small then $\hat{\mathfrak{U}}^*(\rho_2) \ll \hat{\mathfrak{U}}(\rho_2)$. For suppose this is false. Say that 2) fails for $\hat{\mathfrak{U}}^*(\rho_2)$ and $\hat{\mathfrak{U}}(\rho_2)$ when $0 < \rho_2 \leqslant \rho_1$. Hence $\Phi_\nu^{-1}(U^*_{\iota_0 \ldots \iota_\lambda} \times E^n(\rho_2)) - \Phi_\mu^{-1}(U_{\iota_0 \ldots \iota_\lambda} \times E^n(\rho_2))$ are non empty for suitable indices while $\rho_2 \to 0$. Choose a point x_t from each of these sets. Because $x_t \in X(\rho_1)$ which is relatively compact we may assume that $x_t \to x_0$. Obviously we get $x_0 \in \overline{U^*_{\iota_0 \ldots \iota_\lambda}} - U_{\iota_0 \ldots \iota_\lambda}$, a contradiction because $\overline{U^*_{\iota_0\cdots\iota_\lambda}} \subset \overline{U^*_{\iota_0}} \cap \dots \cap \overline{U^*_{\iota_\lambda}} \subset U_{\iota_0\cdots\iota_\lambda}$. In the same way we can prove that condition 3) is satisfied if ρ_2 is sufficiently small and the theorem is clear.

General Theory

Let G be an analytic manifold. We put $\hat{G} = G \times E^n(\rho_1)$ where ρ_1 is an n-tuple of positive numbers. Let $\pi\colon \hat{G} \to E^n(\rho_1)$ and $\mathfrak{P}\colon \hat{G} \to G$ be the projection maps. $\hat{G}^* \subset \hat{G}$ denotes an open subset and $G^* = \hat{G}^* \cap G \times \{0\}$.

The set G^* can be identified with an open subset of G. We denote by $\alpha\colon G^* \times E^n(\rho_1) \to \hat{G}^*$ a biholomorphic fiber preserving map, i.e. $\pi \circ \alpha = \pi^*$ where $\pi^*\colon G^* \times E^n(\rho_1) \to E^n(\rho_1)$ is the natural projection. Let $\rho \leqslant \rho_2 = \gamma\rho_1 < \rho_1$ where $0 < \gamma < 1$ is a fixed number. We put $\hat{G}(\rho) = G \times E^n(\rho)$. If f is a holomorphic function on $\hat{G}(\rho)$ we write $f = \sum a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. We define the norm $\|f\|_\rho$ of f by $\|f\|_\rho = \sup\limits_\nu \{\sup |a_\nu(G)|\}$.

If $f \in I(\hat{G}(\rho))$ we see that $f \circ \alpha$ is a well defined function on $G^* \times E^n(\rho)$ because α is fiber preserving. We define $\|f \circ \alpha\|_\rho$ using G^* instead of G as above. We have the proposition:

Proposition 1. There exists a constant K such that $\|f \circ \alpha\|_\rho \leqslant K\|f\|_\rho$ where $K = K(\rho_2)$ is independent of $\rho \leqslant \rho_2$.

Proof. We write $f = \sum\limits_{|\nu|=0}^{\infty} a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. Now we get $f \circ \alpha = \sum (a_\nu \circ \mathfrak{P} \circ \alpha)(t/\rho)^\nu$ because α is fiber preserving. Since $\mathfrak{P}(\hat{G}^*) \subset G$ we get $|a_\nu \circ \mathfrak{P}(\hat{G}^*)| \leqslant |a_\nu(G)| \leqslant \|f\|_\rho$. Now $a_\nu \circ \mathfrak{P} \circ \alpha$ admits a Taylor series: $a_\nu \circ \mathfrak{P} \circ \alpha = \sum C_{\nu\lambda}(t/\rho)^\lambda$ with $C_{\nu\lambda} \in I(G^*)$. Since $|\sum C_{\nu\lambda}(t/\rho)^\lambda| \leqslant \|f\|_\rho$ in $G^* \times E^n(\rho_1)$ and $\rho \leqslant \rho_2 = \gamma\rho_1$ Cauchy's inequalities give us $|C_{\nu\lambda}(G^*)| \leqslant \|f\|_\rho \gamma^{|\lambda|}$. Let us put $b_\mu = \sum\limits_{\nu+\lambda=\mu} C_{\nu\lambda}$. We get $|b_\mu(G^*)| \leqslant \|f\|_\rho \sum \gamma^{|\lambda|} = \|f\|_\rho (1-\gamma)^{-n} = K\|f\|_\rho$. Now we can write $f \circ \alpha = \sum\limits_\nu a_\nu \circ \mathfrak{P} \circ \alpha (t/\rho)^\nu = \sum\limits_{\lambda,\nu} C_{\nu\lambda}(t/\rho)^\lambda (t/\rho)^\nu = \sum\limits_\mu b_\mu (t/\rho)^\mu$. By definition we have $\|f \circ \alpha\|_\rho = \sup\limits_\mu |b_\mu(G^*)| \leqslant K\|f\|_\rho$.

Let us now consider $\mathbf{h} = (h_{\nu\mu})$ which is a $q \times q$ matrix with $h_{\nu\mu} \in I(\hat{G})$. The $h_{\nu\mu}$ are also assumed to be bounded on $\hat{G}$.