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Autor: Grauert, H.

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The set $G^* \subset G$ is open and $R^{**} = \{ V_1, \ldots, V_{\iota^*} \}$ an open covering of G^* such that $V_\iota \subset\subset U_\iota$ for $\iota \in \{ 1, \ldots, \iota^* \}$. We have:

Cartan's Theorem. There exists a constant K such that if $\xi \in Z^l(R^*, q\mathcal{O})$ then $\xi \mid R^{**} = \delta\eta$ where $\eta \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\| \eta \| \leqslant K \| \xi \|$ for $l \geqslant 1$.

This is a simple consequence of Theorem B and Banach's open mapping theorem.

Now we apply Cartan's theorem. We keep the notations as above. Let $\hat{G} = G \times E^n(\rho)$ and put $\hat{R}^* = \{ U_\iota \times E^n(\rho) \}$. Now $\hat{R}^*$ is a Stein covering of $\hat{G}$. Let $\hat{G}^* = G^* \times E^n(\rho)$ and $\hat{R}^{**} = \{ V_\iota \times E^n(\rho) \}$. Let $\hat{\xi} \in Z^l(\hat{R}^*, q\mathcal{O})$ and write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$ with $\xi_{(\nu)} \in Z^l(R^*, q\mathcal{O})$. We assume $\| \hat{\xi} \|_\rho = \sup\limits_\nu \| \xi_{(\nu)} \| < \infty$. Now Cartan's theorem gives $\xi_{(\nu)} \mid R^{**} = \delta\eta_\nu$ with $\eta_\nu \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\| \eta_\nu \| \leqslant K \| \xi_{(\nu)} \| < \infty$. It follows that $\hat{\eta} = \sum \eta_\nu (t/\rho)^\nu$ is well defined in $C^{l-1}(\hat{R}^{**}, q\mathcal{O})$ and by definition we have $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$.

Smoothing

We are given a sequence of admissible refinements of measure coverings in $X(\rho_1)$. Here $\rho_1 < \rho_0 = \min \rho_\iota$ as usual. Let l be a fixed integer $\geqslant 1$. We are given $\mathfrak{V}^* \ll \mathfrak{V}' = \mathfrak{V}_{3l} \ll \mathfrak{V}_{3l-1} \ll \ldots \ll \mathfrak{V}_1 \ll \mathfrak{V}_0 \ll \mathfrak{V} \ll \mathfrak{U}^* \ll \mathfrak{U} = \mathfrak{U}_{3l} \ll \ldots \ll \mathfrak{U}_0 \ll \mathfrak{U}'$. Here it is also required that $(\mathfrak{V}_{\nu+1}, \mathfrak{U}_{\nu+1}) \ll (\mathfrak{V}_\nu, \mathfrak{U}_\nu)$; $(\mathfrak{V}^*, \mathfrak{U}^*) \ll (\mathfrak{V}', \mathfrak{U})$ and $(\mathfrak{V}_0, \mathfrak{U}_0) \ll (\mathfrak{V}, \mathfrak{U}')$. These extra conditions mean: 1) $\hat{U}^{(\nu+1)}_{i_0 \ldots i_\kappa} \cap \hat{V}^{(\nu+1)}_{\iota_0 \ldots \iota_l} \subset (U^{(\nu)}_{i_0 \ldots i_\kappa} \cap V^{(\nu)}_{\iota_0 \ldots \iota_l})_i$ for each $i \in \{ i_0, \ldots, i_\kappa \}$ and 2) $(U^{(\nu+1)}_{i_0 \ldots i_\kappa} \cap V^{(\nu+1)}_{\iota_0 \ldots \iota_l})_j \subset (U^{(\nu)}_{i_0 \ldots i_\kappa} \cap V^{(\nu)}_{\iota_0 \ldots \iota_l})_i$ for all $i, j \in \{ i_0, \ldots i_\kappa, \iota_0, \ldots \iota_l \}$. Recall that all operations are done with respect to ρ_1. Let us put $\hat{R}^{(\nu)}_{i_0 \ldots i_k \iota_0 \ldots \iota_\kappa} = \hat{U}^{(\nu)}_{i_0 \ldots i_k} \cap \hat{V}^{(\nu)}_{\iota_0 \ldots \iota_\kappa}$. We consider elements $\xi_{i_0 \ldots i_k \iota_0 \ldots \iota_\kappa} \in \hat{\Gamma}(\hat{R}^{(\nu)}_{i_0 \ldots i_k \iota_0 \ldots \iota_\kappa}, \mathbf{F})$. Now we take a full collection $\hat{\xi} = \{ \hat{\xi}_{i_0 \ldots i_k \iota_0 \ldots \iota_\kappa} \}$ of such elements which is anticommutative in $\{ i_0, \ldots i_k \}$ and $\{ \iota_0, \ldots, \iota_\kappa \}$. In this way we get a double complex $C_\nu^{k,\kappa}$. Here $\delta : C_\nu^{k,\kappa} \to C_\nu^{k+1,\kappa}$ and $\partial : C_\nu^{k,\kappa} \to C_\nu^{k,\kappa+1}$ are the usual coboundary operators.

Norm in $C_\nu^{k,\kappa}$: Let $\hat{\xi} \in C_\nu^{k,\kappa}$; we put

$\| \hat{\xi} \|_\rho = \max_{i,(i_0,\ldots i_k,\iota_0,\ldots,\iota_\kappa)} \{ \| \hat{\xi}_{i_0\ldots i_k \iota_0 \cdots \iota_\kappa} | (R^{(\nu+1)}_{i_0\ldots i_k \iota_0\ldots\iota_\kappa})_i(\rho) \|_i$ with $i \in \{i_0,\ldots, i_k\}\}$. Here $\rho \geqslant \rho_1$ and $R^{(\nu+1)}_{i_0\ldots i_k, \iota_0\ldots\iota_\kappa} = U^{(\nu+1)}_{i_0\ldots i_k} \cap V^{(\nu+1)}_{\iota_0\ldots\iota_\kappa}$ and $\| \, \|_i$ is taken with respect to the chart $\mathscr{W}_i$ as usual.

Smoothing Lemma: Let $\kappa > 0$. There exists a constant K such that: If $\hat{\xi} \in C_\nu^{k,\kappa}$ with $\partial \hat{\xi} = 0$ and $\| \hat{\xi} \|_\rho < \infty$ then we can find $\hat{\eta} \in C^{k,\kappa-1}_{\nu+3}$ such that $\hat{\xi} \mid C^{k,\kappa}_{\nu+3} = \partial \hat{\eta}$ and $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Here $\rho \leqslant \rho_2 = \gamma\rho_1$ with $0 < \gamma < 1$ and K depends only on ρ_2.

Proof. Let us fix $i_0, \ldots, i_k$ in the following discussion. Let $G = U^{(\nu+1)}_{i_0\ldots i_k}$ and put $\hat{G} = (G)_i(\rho_1)$ for some $i \in \{ i_0, \ldots, i_k \}$ which is also fixed now. Now G is Stein in X_0 and $\hat{G}$ is Stein in X. We put $R^* = G \cap \mathfrak{V}_{\nu+1}$ which is a Stein covering of G. Also $\hat{R}^* = \{ (G \cap V_\iota^{(\nu+1)})_i(\rho_1) \}_{\iota=1,\ldots,\iota^*}$ is a Stein covering of $\hat{G}$. Let $\hat{\xi} = \{ \hat{\xi}_{i_0,\ldots i_k,\iota_0\ldots\iota_\kappa} \}$. Now we look at the elements of $\{ \hat{\xi}_{i_0,\ldots i_k,\iota_0\ldots\iota_\kappa} \} = \hat{\xi}_{i_0,\ldots i_k} \in Z^\kappa(\hat{R}^*, \mathbf{F})$. Here $i_0, \ldots i_k$ is fixed as above. We get a cocycle because we have assumed that $\partial \hat{\xi} = 0$. More precisely we have considered the restriction of $\hat{\xi}_{i_0, \ldots i_k, \iota_0,\ldots,\iota_\kappa}$ to $\hat{R}^*$. We must verify that this restriction is possible.

Verification: By definition of $Z^\kappa(\hat{R}^*, \mathbf{F})$ we have to look at sets of the following type: (these are the sets where the cross-sections are defined) $(G \cap V^{(\nu+1)}_{\iota_0})_i \cap \ldots \cap (G \cap V^{(\nu+1)}_{\iota_\kappa})_i = (G \cap V^{(\nu+1)}_{\iota_0\ldots\iota_\kappa})_i = (R^{(\nu+1)}_{i_0\ldots i_k \iota_0\ldots\iota_\kappa})_i$. Now by 2) we have $(R^{(\nu+1)}_{i_0\ldots i_k \iota_0\cdots\iota_\kappa})_i \subset \bigcap_j (R^{(\nu)}_{i_0\cdots i_k \iota_0\cdots\iota_\kappa})_j \subset (U^{(\nu)}_{i_0})_{i_0} \cap \ldots \cap (V^{(\nu)}_{\iota_\kappa})_{\iota_\kappa} = \hat{R}^{(\nu)}_{i_0\cdots i_k \iota_0\cdots\iota_\kappa}$. Q.E.D.

Now we put $G^* = U^{(\nu+2)}_{i_0\ldots i_k} \subset\subset G$. We let $\hat{R}^{**} = \{(G^* \cap V_\iota^{(\nu+2)})_i\}_{\iota=1,\ldots,\iota^*}$. The system $\hat{R}^{**}$ is a Stein covering of $(G^*)_i$. We are in a good position now. For we are given $\hat{\xi}_{i_0,\ldots i_k} \in Z^\kappa(\hat{R}^*, \mathbf{F})$. Here $\hat{R}^*$ is a Stein covering of $\hat{G}$ and $\hat{G}$ is a Stein manifold. We are working in the chart $\mathscr{W}_i$ where the usual identifications are used. Hence we arrive at the following situation: G is a Stein manifold with a Stein covering $R^* = \mathfrak{V}_{\nu+1} \cap G$. Also $G^* \subset\subset G$ and $R^{**} = \mathfrak{V}_{\nu+2} \cap G^*$ is a Stein covering of G^* such that $R^{**} \subset\subset R^*$. The cocycle $\hat{\xi}_{i_0,\ldots i_k}$ is now considered as an element of $Z^\kappa(\hat{R}^*, q\mathcal{O})$ which we simply call $\hat{\xi}_{i_0 \dots i_k}$ again. Now we apply the result after Cartan's theorem. Hence we can find a constant K such that for every $\rho \leqslant \rho_2$ we get $\eta \in$ $\in C^{\kappa-1}(\hat{R}^{**}, q\mathcal{O})$ and $\| \eta \|_\rho \leqslant K \| \hat{\xi}_{i_0, \dots i_k} \|_\rho$ with $\partial \eta = \hat{\xi}_{i_0 \dots i_k}$. But this means precisely that we can find $\hat{\eta}_{i_0, \dots i_k} \in C^{\kappa-1}(\hat{R}^{**}(\rho), \mathbf{F})$ such that $\| \hat{\eta}_{i_0 \dots i_k} \|_{i,\rho} \leqslant K \| \hat{\xi}_{i_0 \dots i_k} \|_{i,\rho}$ with $\hat{\xi}_{i_0 \dots i_k} = \partial \hat{\eta}_{i_0 \dots i_k}$. We have only constructed $\hat{\eta}_{i_0 \dots i_k}$ using a fixed $i \in \{ i_0, \dots, i_k \}$. Now we must let $(i_0, \dots, i_k)$ vary. For each $(i_0, \dots i_k)$ we choose some i which only depends on the unordered $(k+1)$-tupel $(i_0, \dots, i_k)$ and construct an element $\hat{\eta}_{i_0, \dots i_k}$ as above. Now we can restrict everything to $C^{k,\kappa-1}_{\nu+3}$.

Verification: Consider a set where cross-sections over $C^{k,\kappa-1}_{\nu+3}$ have to be defined, i.e. a set $\hat{U}^{(\nu+3)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu+3)}_{\iota_0 \dots \iota_\kappa}$. But by 1) follows $\hat{U}^{(\nu+3)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu+3)}_{\iota_0 \dots \iota_\kappa} \subset$ $\subset (R^{(\nu+2)}_{i_0 \dots i_k, \iota_0 \dots \iota_\kappa})_i$ for each $i \in \{ i_0, \dots, i_\kappa \}$. This inclusion shows that we get a well defined element $\hat{\eta} \in C^{k,\kappa-1}_{\nu+3}$ by restricting the elements $\hat{\eta}_{i_0, \dots i_k}$ to $C^{k,\kappa-1}_{\nu+3}$. We find that $\hat{\xi} \mid C^{k,\kappa}_{\nu+3} = \partial \hat{\eta}$ now. The norm inequalities are not obvious, but recalling how $\hat{\eta}$ is constructed here it is seen that we can apply Theorem I to obtain the required estimate.

SMOOTHING THEOREM. There exists a constant K such that: If $\hat{\xi} \in$ $\in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ then we can find $\hat{\xi}^* \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}'(\rho). \mathbf{F})$ for which $\hat{\xi}^* \mid \hat{\mathfrak{B}}'(\rho) = \hat{\xi} \mid \hat{\mathfrak{B}}'(\rho) + \delta \hat{\eta}$ and $\| \hat{\xi}^* \|_\rho$ and $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Here $\rho \leqslant \rho_2 < \rho_1$ and K only depends on ρ_2.

Proof. Before we can use the double complex $\{ C^{k,\kappa}_\nu \}$ we must introduce two " ε-maps ". To define the ε_1-map, let $Z^{k,\kappa}_\nu \subset C^{k,\kappa}_\nu$ consist of all $\hat{\xi} \in C^{k,\kappa}_\nu$ such that $\delta \hat{\xi} = \partial \hat{\xi} = 0$. Now we shall define the ε_1-map : ε_1 : $Z^l(\hat{\mathfrak{B}}, \mathbf{F}) \to Z^{0,l}_0$. A section belonging to an element of $C^{0,l}_0$ is defined on some set $\hat{U}^{(0)}_{i_0} \cap \hat{V}^{(0)}_{\iota_0, \dots \iota_l} \subset \hat{V}_{\iota_0 \dots \iota_l}$ where sections of elements of $Z^l(\hat{\mathfrak{B}}, \mathbf{F})$ are defined. Hence we get a natural restriction map ε_1 which also maps cocycles into cocycles. It is easy to verify that $\| \varepsilon_1(\hat{\xi}) \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Theorem I can be used because $(U^{(1)}_i \cap V^{(1)}_{\iota_0 \dots \iota_l})_i \subset (V^{(0)}_{\iota_0 \dots \iota_l})_\iota$ for every i and every $\iota \in \{ \iota_0, \dots \iota_l \}$. Recall that the norm in $Z^l(\hat{\mathfrak{B}}, \mathbf{F})$ is defined with respect to

$\hat{\mathfrak{V}}_0$ here. The " ε_2-map ": we shall construct a map $\varepsilon_2\colon \hat{Z}^{l,0}_{3l} \to Z^l(\mathfrak{U}, \hat{\mathbf{F}})$. Let $\hat{\xi} = \{ \hat{\xi}_{i_0,\dots i_l,\iota_0} \} \in Z^{l,0}_{3l}$. Here $\hat{\xi}_{i_0,\dots i_l,\iota_0}$ is defined on $\hat{R}^{(3l)}_{i_0\dots i_l,\iota_0}$. Because $\partial\hat{\xi} = 0$ we see that the elements $\hat{\xi}_{i_0\dots i_l,\iota_0}$ are independent of ι_0. Now $\bigcup_{\iota=1}^{\iota^*} \hat{V}^{(3l)}_\iota$ covers $X(\rho_1)$. If we put $\varepsilon_2(\hat{\xi})_{i_0\dots i_l} = \hat{\xi}_{i_0\dots i_l,\iota_0}$ in $\hat{U}^{(3l)}_{i_0\dots i_l} \cap \hat{V}^{(3l)}_{\iota_0}$ then we see that $\varepsilon_2(\hat{\xi})_{i_0\dots i_l}$ is a well defined section on $\hat{U}^{(3l)}_{i_0\dots i_l}$. In this way we obtain $\varepsilon_2(\hat{\xi}) \in Z^l(\hat{\mathfrak{U}}, \hat{\mathbf{F}})$. Here $\varepsilon_2(\hat{\xi})$ is a cocycle because $\delta\hat{\xi} = 0$. Now we prove that $\| \varepsilon_2(\hat{\xi}) \|_\rho \leqslant K \| \hat{\xi} \|_\rho$.

Verification. A computation of $\| \varepsilon_2(\hat{\xi}) \|_\rho$ involves the following: $\varepsilon_2(\hat{\xi}) = \{ \xi^{(2)}_{i_0\dots i_l} \}$. Look at some $\xi^{(2)}_{i_0\dots i_l}$ in the chart $\mathscr{W}_i$ with $i \in \{ i_0, \dots, i_l \}$. We write $\hat{\xi}^{(2)}_{i_0\dots i_l} = \sum a_\nu (t/\rho)^\nu$ over $(U^*_{i_0}\dots_{i_l})_i$ and compute $\sup_\nu | a_\nu (U^*_{i_0}\dots i_l) |$. A computation of $\| \hat{\xi} \|_\rho$ involves the following: Look at $\hat{\xi}_{i_0\dots i_l}$ over $(U^*_{i_0}\dots_{i_l} \cap \cap V^*_\iota)_i$ in a chart W_i. Here ι is fixed. We write $\hat{\xi}_{i_0\dots i_l,\iota} = \sum a_\nu^{(\iota)} (t/\rho)^\nu$ and compute $\sup_\nu | a_\nu^{(\iota)} (U^*_{i_0},\dots_{i_l} \cap V^*_\iota) |$. Now $\bigcup_1^{\iota^*} V^*_\iota$ covers X_0. Hence we would have $\sup_{\nu,\iota} | a_\nu^{(\iota)}(U^*_{i_0}\dots_{i_l} \cap V^*_\iota) | = \sup_\nu | a_\nu (U^*_{i_0}\dots_{i_l}) |$ if $a_\nu = a_\nu^{(\iota)}$ in $U^*_{i_0}\dots_{i_l} \cap \cap V^*_\iota$. But this is obvious since $\xi^{(2)}_{i_0}\dots_{i_l} = \hat{\xi}_{i_0\dots i_l,\iota}$ in $(U^*_{i_0}\dots_{i_l} \cap V^*_\iota)_i$. Hence we have $\| \varepsilon_2(\hat{\xi}) \|_\rho \leqslant \| \hat{\xi} \|_\rho$.

Now we are ready to start the proof of the smoothing theorem. We let K denote a constant, which may be different at different occurences. We also introduce a double complex $\{ \tilde{C}^{k,\kappa}_\nu \}$ using $(\mathfrak{V}, \mathfrak{V})$, i.e. it is defined just as the previous double complex was, using $\mathfrak{V}$-sets instead of $\mathfrak{U}$-sets. We shall inductively construct the following elements:

$$\hat{\xi}_\nu = \{ \hat{\xi}_{i_0\dots i_\nu,\iota_0\dots\iota_{l-\nu}} \} \in Z^{\nu,l-\nu}_{3\nu}$$

$$\tilde{\xi}_\nu = \{ \tilde{\xi}_{i_0\dots i_\nu,\iota_0\dots\iota_{l-\nu}} \} \in \tilde{Z}^{\nu,l-\nu}_{3\nu};\ \nu = 0,\dots,l$$

$$\hat{\eta}_\nu = \{ \hat{\eta}_{i_0\dots i_{\nu-1},\iota_0\dots\iota_{l-\nu}} \} \in C^{\nu-1,l-\nu}_{3\nu}$$

$$\tilde{\eta}_\nu = \{ \tilde{\eta}_{i_0\dots i_{\nu-1},\iota_0\dots\iota_{l-\nu}} \} \in \tilde{C}^{\nu-1,\nu-1}_{3\nu};\quad \nu = 1,\dots,l$$

$$\tilde{\gamma}_\nu = \{ \tilde{\gamma}_{i_0 \ldots i_{\nu-1}, \iota_0 \ldots \iota_{l-\nu-1}} \} \in \tilde{C}_{3\nu-3}^{\nu-1, l-\nu-1}; \quad \nu = 1, \ldots, (l-1)$$

and $\tilde{\gamma}_l = \{ \tilde{\gamma}_{i_0 \ldots i_{l-1}} \} \in C^{l-1}(\mathfrak{V}_{3l})$.

The construction: $\hat{\xi} \in Z^l(\hat{\mathfrak{V}}(\rho), \mathbf{F})$ is given. The whole construction is done using ρ instead of ρ_1 and we omit ρ to simplify the notation. We put $\varepsilon_1(\hat{\xi}) = \hat{\xi}_0 \in Z_0^{0,l}$. Now we apply the Smoothing Lemma and get $\hat{\eta}_1$ such that $\partial \hat{\eta}_1 = \hat{\xi}_0$ with $\| \hat{\eta}_1 \|_\rho \leqslant K \| \hat{\xi}_0 \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Put $\hat{\xi}_1 = \delta \hat{\eta}_1$. Obviously $\| \hat{\xi}_1 \|_\rho \leqslant K \| \hat{\eta}_1 \|_\rho$. Inductively we find $\delta \hat{\eta}_\nu = \hat{\xi}_{\nu-1}$ and we put $\hat{\xi}_\nu = \delta \hat{\eta}_\nu$ where $\hat{\eta}_\nu$ are found from the Smoothing Lemma. Finally we get $\hat{\xi}_l$ and we have $\| \hat{\xi}_l \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Now we define $\tilde{\xi}_\nu$ and $\tilde{\eta}_\nu$ as follows. Put $\tilde{\xi}_0 = \hat{\xi}_0$ where $\tilde{\xi}_0 \in \tilde{Z}_0^{0,l}$ is obtained by natural restriction of $\hat{\xi}_0$. Put $\tilde{\eta}_\nu = (-1)^\nu \{ \hat{\xi}_{i_0 \ldots i_{\nu-1}, \iota_0 \ldots \iota_{l-\nu}} \}$ which is well defined with respect to $(\mathfrak{V}_{3\nu}, \mathfrak{V}_{3\nu})$ by taking natural restrictions. Put $\tilde{\xi}_\nu = \delta \tilde{\eta}_\nu$ for $\nu = 1, \ldots, l$. A computation shows that $\tilde{\xi}_{\nu-1} = \partial \tilde{\eta}_\nu$ when $\nu = 1, \ldots, l$. Notice that this is trivial when $\nu = 1$. In the following discussion each $\hat{\eta}_\nu$ is restricted to $(\mathfrak{V}_{3\nu}, \mathfrak{V}_{3\nu})$. We have $\partial(\tilde{\eta}_1 - \hat{\eta}_1) = 0$. Hence we find $\tilde{\eta}_1 - \hat{\eta}_1 = \partial \tilde{\gamma}_1$ by the Smoothing Lemma. Now we define $\tilde{\gamma}_\nu$ such that $\partial \tilde{\gamma}_\nu = \tilde{\eta}_\nu - \hat{\eta}_\nu - \delta \tilde{\gamma}_{\nu-1}$ inductively. This is possible because $\partial(\tilde{\eta}_\nu - \hat{\eta}_\nu - \delta \tilde{\gamma}_{\nu-1}) = 0$, for we have $\partial(\tilde{\eta}_\nu - \hat{\eta}_\nu - \delta \tilde{\gamma}_{\nu-1}) = \tilde{\xi}_{\nu-1} - \hat{\xi}_{\nu-1} - \delta \partial \tilde{\gamma}_{\nu-1} = \delta \tilde{\eta}_{\nu-1} - \delta \hat{\eta}_{\nu-1} - \delta(\tilde{\eta}_{\nu-1} - \hat{\eta}_{\nu-1}) = 0$. We get finally $\tilde{\gamma}_{l-1} \in \tilde{C}_{3l}^{l-2,0}$ and then $\delta \tilde{\gamma}_{l-1} \in \tilde{C}_{3l}^{l-1,0}$. We have $\partial(\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1}) = 0$. Therefore we can put $\tilde{\gamma}_l = \varepsilon_2(\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1})$. It follows that $\tilde{\gamma}_l \in C^{l-1}(\mathfrak{V}_{3l})$ and $\delta \tilde{\gamma}_l = \varepsilon_2(\tilde{\xi}_l - \hat{\xi}_l)$. We have $\varepsilon_2(\tilde{\xi}_l) = -\hat{\xi} \mid \mathfrak{V}'$ and for $\varepsilon_2(\hat{\xi}_l) = -\hat{\xi}^*$ and $\hat{\eta} = \tilde{\gamma}_l$ the required equation $\hat{\xi}^* = \hat{\xi} + \delta \hat{\eta}$. The estimates follow immediately from the construction and the Smoothing Lemma.