

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COHERENCE OF DIRECT IMAGES
Autor: Grauert, H.
Kapitel: Approximation
DOI: <https://doi.org/10.5169/seals-42345>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 02.02.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

APPROXIMATION

We use positive n -tuples ρ, \dots with $\rho \leq \rho_2 < \rho_3 < \rho_4 < \rho_1$ and $\rho = \gamma'' \rho_1, \rho_2 = \gamma \rho_1, \rho_3 = \gamma' \rho_1, \rho_4 = \gamma''' \rho_1$. The n -tuple ρ_1 is defined as in the smoothing theorem.

Definition: $H_*^l = \{ \xi \in H^l(X_0, \underline{F|X_0}) \text{ such that there exists } U = U(0) \text{ in } E^n \text{ with } \hat{\xi} \in H^l(\psi^{-1}(U), \mathbf{F}) \text{ and } \hat{\xi}|X_0 = \xi \}$. Serre's theorem gives $\dim_{\mathbf{C}} H_*^l \leq \dim_{\mathbf{C}} H^l(X_0, \underline{F|X_0}) < \infty$. In the following discussion we are given $\hat{b}_1, \dots, \hat{b}_r$ in $Z^l(\hat{\mathcal{U}}'(\rho_4), \mathbf{F})$ such that $\hat{b}_1|X_0, \dots, \hat{b}_r|X_0$ constitute a base of the complex vector space H_*^l . For this to be possible, ρ_4 has to be chosen small enough. Here $\hat{\mathcal{U}}'$ is a Stein covering of $X(\rho_1)$ and defined as in the smoothing theorem. We also assume that we are given a sequence of measure coverings as there. Further we construct the sequence so that there are still sufficiently many measure coverings in between \mathfrak{B} and \mathcal{U} . These are denoted by \mathcal{U}_v^* . We have $\mathcal{U} \gg \mathcal{U}_1^* \gg \mathcal{U}_2^* \gg \dots \gg \mathfrak{B}$. The n -tuple ρ_3 is also fixed from now on and K always denotes (possibly different) constants.

Approximation Lemma: Let $\varepsilon > 0$. Then we can find ρ_2 such that: If $\rho \leq \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$ with $\|\hat{\xi}\|_{\rho} < \infty$ (the norm is taken with respect to $\hat{\mathcal{U}}_1^*(\rho)$), then there exist $a_1, \dots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ such that $\tilde{\xi} = \hat{\xi} - \sum_1^r a_i \hat{b}_i - \delta \hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$. Here $\tilde{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ and $\|\tilde{\xi}\|_{\rho} \leq \varepsilon \|\hat{\xi}\|_{\rho}$ and $\|a_v\|_{\rho}, \|\hat{\eta}\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$. K is a fixed constant.

Proof. We shall first prove some results which are needed later on. Let $S \in \Gamma(\hat{U}_{\iota_0 \dots \iota_l}(\rho), \mathbf{F})$. Choose $\iota \in \{ \iota_0, \dots, \iota_l \}$. Now $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota} \subset \hat{U}_{\iota_0 \dots \iota_l}$ because $\mathcal{U}_1^* \ll \mathcal{U}$. The operations are always defined with respect to ρ_1 . We can now restrict S to $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$. In the chart \mathcal{W}_{ι} we can write $S = \sum a_v (t/\rho)^v$. Here $a_v \in qI(U_{\iota_0 \dots \iota_l}^{(1)*})$. Now the a_v are extended constantly and we get elements $\hat{a}_v \in \Gamma((U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}, \mathbf{F})$. Let us put $S_v = \hat{a}_v | \hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$. We claim that $\|S_v\|_{\rho_1} \leq K \|S\|_{\rho}$. For obviously $\|S\|_{\rho} \geq |a_v (U_{\iota_0 \dots \iota_l}^{(1)*})|$ and

we can use the Theorem I to prove that $\|S_v\|_{\rho_1} \leq K \|\hat{a}_v\| (U_{\iota_0 \dots \iota_l}^{(1)*}(\rho_1))\|_{\iota} = K \|a_v(U_{\iota_0 \dots \iota_l}^{(1)*})\| \leq K \|S\|_{\rho}$. Q.E.D.

Let S'_v be defined using some other $\iota' \in \{\iota_0, \dots, \iota_l\}$. Then $S_v - S'_v \in \Gamma(\hat{U}_{\iota_0 \dots \iota_l}^{(2)*}, \mathbf{F})$. We claim that $\|S_v - S'_v\|_{\rho_4} \leq K \gamma''' \|S\|_{\rho}$.

Proof. Define $\alpha_s = \sum_{|\lambda|=s} a_{\lambda}(t/\rho)^{\lambda}$ and $\beta_s = \sum_{|\lambda|=0}^{s-1} a_{\lambda}(t/\rho)^{\lambda}$ over $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$. We do the same for ι' respectively and obtain α'_s and β'_s over $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota'}(\rho)$. For the restrictions to $\hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$ we see that $\alpha_s - \alpha'_s = -(\beta_s - \beta'_s)$. Hence we get $\|\alpha_s - \alpha'_s\|_{\rho_4} \leq K(\gamma''')^s \|\alpha_s - \alpha'_s\|_{\rho_1} = K(\gamma''')^s \|\beta_s - \beta'_s\|_{\rho_1} \leq K(\gamma''')^s \|\beta_s\|_{\rho_1} + K(\gamma''')^s \|\beta'_s\|_{\rho_1} \leq K(\gamma''')^s [\|\beta_s\|_{\rho_1}^* + \|\beta'_s\|_{\rho_1}^*] \leq K(\gamma''')^s (\gamma'')^{1-s} \|S\|_{\rho}$. Here the norms are defined with respect to $U_{\iota_0 \dots \iota_l}^{(3)*}$ except $\|\cdot\|_{\rho_1}^*$ and $\|S\|_{\rho}$ which are defined with respect to $U_{\iota_0 \dots \iota_l}^{(1)*}$. Now we look at the difference $(S_v - S'_v) t^v/\rho^v$ on $(U_{\iota_0 \dots \iota_l}^{(3)*})_{\mu}$ with $|v|=s$, $\mu \in \{\iota_0, \dots, \iota_l\}$, and the power series development with respect to W_{μ} . There is one term of order s which is equal to the corresponding term of $\alpha_s - \alpha'_s$. Therefore its norm is $\leq K(\gamma''')^s \cdot (\gamma'')^{1-s} \|S\|_{\rho}$. Moreover we have $\|S_v(t/\rho)^v - S'_v(t/\rho)^v\|_{\rho_1} \leq (\gamma'')^{-s} \cdot K \|S\|_{\rho}$ where the first norm is defined with respect to $U_{\iota_0 \dots \iota_l}^{(3)*}$. For the sum \sum of terms of higher order than s in the power series of $(S_v - S'_v) t^v/\rho^v$ we therefore get: $\|\sum\|_{\rho_4} \leq (\gamma''')^{s+1} (\gamma'')^{-s} \cdot K \|S\|_{\rho}$. Hence we get $\|(S_v - S'_v)\|_{\rho_4} \leq \gamma''' \cdot K \|S\|_{\rho}$. This proves our statement. We see that K is independent of ρ_4 and S . The number γ''' depends on ρ_4 only, so $\gamma''' \cdot K$ gets very small if we make ρ_4 very small.

Let $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$ with $\hat{\xi} = \{\hat{\xi}_{\iota_0 \dots \iota_l}\}$. Choose $\iota = \iota(\iota_0, \dots, \iota_l)$ as a function of the unordered $(l+1)$ -tuple. We now fix ι_0, \dots, ι_l and write $S = \hat{\xi}_{\iota_0 \dots \iota_l}$. We apply to S the method described above and obtain $\hat{\xi}_{\iota_0 \dots \iota_l}^{(v)} = S_v$. We do this now for every ι_0, \dots, ι_l and consider $\hat{\xi}_{(v)} = \{\hat{\xi}_{\iota_0 \dots \iota_l}^{(v)}\}$ as an element of $C^l(\hat{\mathcal{U}}_2^*(\rho_4), \mathbf{F})$. Of course $\hat{\xi}_{(v)}$ depends on the choice of $\iota = \iota(\iota_0 \dots \iota_l)$ here. Now we see that $\|\hat{\xi}_{(v)}\|_{\rho_4} \leq \|\hat{\xi}_{(v)}\|_{\rho_1} \leq K \|\hat{\xi}\|_{\rho}$. We also wish to estimate $\delta \hat{\xi}_{(v)}$. Because $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$ we can use the preliminary result on ι and ι' to obtain $\|\delta \hat{\xi}_{(v)}\|_{\rho_4} \leq K \gamma''' \|\hat{\xi}\|_{\rho}$.

We shall also need another result:

Induction Lemma: There exists $\hat{\eta}_v \in C^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$ such that $\delta\hat{\eta}_v = \delta\hat{\xi}_{(v)}$ on $\hat{\mathcal{U}}_4^*(\rho_3)$ and $\|\hat{\eta}_v\|_{\rho_3} \leq K \|\delta\hat{\xi}_{(v)}\|_{\rho_4}$.

Proof. The proof uses the assumption that $\psi_{(l+1)}(\mathbf{F})$ is coherent. Because the coherence of direct images is proved by downward induction on l , this assumption can be made. Moreover it is assumed that the main theorem is proved for dimension $l + 1$ already. Let us now put $\alpha = \delta\hat{\xi}_{(v)} \in B^{l+1}(\hat{\mathcal{U}}_2^*(\rho_4), \mathbf{F})$ and $\hat{\eta}_v = \beta \in C^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$. We have to prove the existence of β . We may assume that ρ_4 is so small that the main theorem is valid for $\rho \leq \rho_4$ in the case of dimension $l + 1$. So there are cocycles $\omega_1, \dots, \omega_r \in Z^{l+1}(\hat{\mathcal{U}}(\rho_4), \mathbf{F})$ such that $\alpha = \sum C_\lambda \omega_\lambda + \delta\eta$, where $C_\lambda \in I(E^n(\rho_4))$ and $\eta \in C^l(\hat{\mathcal{U}}_4^*(\rho_4), \mathbf{F})$. We have to assume that between $\hat{\mathcal{U}}_4^*$ and \mathcal{U}_2^* there are very many measure coverings. The cross-sections $\psi_{(l+1)}(\omega_\lambda)$ give a homomorphism $r\mathcal{O} \rightarrow \psi_{(l+1)}(\mathbf{F})$ over $E^n(\rho_4)$. Because $\psi_{(l+1)}(\mathbf{F})$ is coherent the kernel \mathcal{N} is coherent again. Over $E^n(\rho')$ with $\rho_3 < \rho' < \rho_4$ we find an epimorphism $p\mathcal{O} \rightarrow \mathcal{N}$. Denote by n_1, \dots, n_p the images of the unit cross-sections in $p\mathcal{O}$. Write $n_\lambda = (e_{\lambda 1}, \dots, e_{\lambda r})$ as an r -tupel of holomorphic functions. The image of n_λ in $\Gamma(E^n(\rho'), \psi_{(l+1)}(\mathbf{F}))$ is $\psi_{(l+1)}(\sum_{\mu=1}^r e_{\lambda\mu} \omega_\mu)$ and zero. We may choose ρ_2 and then ρ_3 and ρ' very small. Then it follows that $\hat{n}_\lambda = \sum e_{\lambda\mu} \omega_\mu$ is a coboundary. If $\rho_3 < \rho'' < \rho'$ there are cochains $\eta_\lambda \in C^l(\hat{\mathcal{U}}_4^*(\rho''), \mathbf{F})$ such that $\delta\eta_\lambda = \hat{n}_\lambda$. Now $(C_1, \dots, C_r) \in \Gamma(E^n(\rho_4), \mathcal{N})$. By the methods of sheaf theory we can lift this cross-section to $p\mathcal{O}$. Using a "Banach open mapping theorem" we see that the map $\Gamma(E^n(\rho'), p\mathcal{O}) \rightarrow \Gamma(E^n(\rho'), \mathcal{N})$ is open. This means here that we can find holomorphic functions a_λ over $E^n(\rho_3)$ such that $C_\mu = \sum a_\lambda e_{\lambda\mu}$ and $\|a_\lambda\|_{\rho_3} \leq K \max_\mu \|C_\mu\|_{\rho'} \leq K \max_\mu \|C_\mu\|_{\rho_4}$. We get $\sum C_\mu \omega_\mu = \sum a_\lambda e_{\lambda\mu} \omega_\mu = \sum a_\lambda \hat{n}_\lambda = \delta(\sum a_\lambda \eta_\lambda)$. This leads to $\alpha \in C^{l+1}(\hat{\mathcal{U}}_4^*(\rho_3)) = \delta(\eta + \sum a_\lambda \eta_\lambda)$. The estimates required obviously hold. Q.E.D.

Let us now put $\hat{\xi}_{(v)}^* = \hat{\xi}_{(v)} - \hat{\eta}_v \in Z^l(\hat{\mathcal{U}}_4(\rho_3), \mathbf{F})$. We can write $\hat{\xi}_{(v)}^* \mid X_0 = \sum a_{v\lambda} \hat{b}_\lambda \mid X_0 + \delta\gamma_v$ over \mathcal{U}_6^* . Here $a_{v\lambda}$ are complex numbers and $\gamma_v \in C^{l-1}(\mathcal{U}_6^*, \mathbf{F} \mid X_0)$. Cartan's theorem and the result after that give the estimates $|a_{v\lambda}| \leq K \|\hat{\xi}_{(v)}^*\|_{\rho_3} \leq K \|\hat{\xi}\|_\rho$ and $\|\gamma_v\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}^*\|_{\rho_3} \leq K \|\hat{\xi}\|_\rho$. Here $\hat{\gamma}_v \in C^{l-1}(\hat{\mathcal{U}}_7^*(\rho_3), \mathbf{F})$ has been obtained by a constant

extension of γ_v . Let us now put $\hat{\xi}_{(v)}^{(1)} = \hat{\xi}_{(v)}^* - \sum a_{v\lambda} \hat{b}_\lambda - \delta \hat{\gamma}_v$. Here $\hat{\xi}_{(v)}^{(1)} \in C^l(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbb{F})$. Using the previous estimates and the fact that the \hat{b}_λ are finite we find that $\|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}\|_{\rho_4} \leq K \|\hat{\xi}\|_{\rho}$.

Now we also have $\hat{\xi}_{(v)}^{(1)}|_{X_0} = 0$. It follows that

$$\|\hat{\xi}_{(v)}^{(1)}\|_{\rho} \leq \gamma/\gamma' \|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}.$$

Finally we put in $\hat{\mathfrak{U}}_9^*(\rho)$:

$$\begin{aligned} \hat{\xi}^{(1)} &= \sum \hat{\xi}_{(v)}^{(1)} (t/\rho)^v = \\ &= \sum \hat{\xi}_{(v)} (t/\rho)^v - \sum \hat{\eta}_v (t/\rho)^v - \sum a_{v\lambda} (t/\rho)^v \hat{b}_\lambda - \delta (\sum \hat{\gamma}_v (t/\rho)^v) \\ &= \hat{\xi} - \hat{\eta} - \sum a_\lambda \hat{b}_\lambda - \delta \hat{\gamma}. \end{aligned}$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of $\hat{\xi}_{(v)}^{(1)}$ by (t/ρ) is smaller than $\gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}$ and that with respect to $\hat{\eta}_v$ is smaller than $\gamma''' \cdot K \|\hat{\xi}\|_{\rho}$ we find: $\|\hat{\xi}^{(1)}\|_{\rho} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}$ and $\|\hat{\eta}\|_{\rho} \leq \gamma''' \cdot K \|\hat{\xi}\|_{\rho}$ and $\|a_\lambda\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$. We take the restriction to $\hat{\mathfrak{B}}(\rho)$ and now $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ is the desired element. Of course we have to choose ρ_4 and then ρ_2 small enough, for example let $\gamma''' < \varepsilon/2 K$ and $\gamma \leq \varepsilon\gamma'/2 K$.

MAIN THEOREM

There exists ρ_2 and a constant K such that if $\rho \leq \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$ with $\|\hat{\xi}\|_{\rho} < \infty$ then we can find $a_1, \dots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ such that $\hat{\xi} = \sum a_\lambda \hat{b}_\lambda + \delta \hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$ with $\|\hat{\eta}\|_{\rho}$ and $\|a_v\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$.

Proof. We have one constant K from the smoothing theorem. Now we find ρ_2 with an ε in the Approximation Lemma such that $\varepsilon \cdot K < 1/2$. We shall use this ρ_2 and prove the theorem here. We are given $\hat{\xi}_0 = \hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$ with $\|\hat{\xi}\|_{\rho} < \infty$. The Approximation Lemma gives $\tilde{\xi}_1 =$