Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COHERENCE OF DIRECT IMAGES

Autor: Grauert, H.

Kapitel: Main Theorem

DOI: https://doi.org/10.5169/seals-42345

extension of γ_ν. Let us now put $\hat{\xi}^{(1)}_{(\nu)} = \hat{\xi}^*_{(\nu)} - \sum a_{\nu\lambda} \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma}_\nu$. Here $\hat{\xi}^{(1)}_{(\nu)} \in$ $\in C^l(\hat{\mathfrak{U}}^*_7(\rho_3), \mathbf{F})$. Using the previous estimates and the fact that the $\hat{\mathfrak{b}}_\lambda$ are finite we find that $\| \hat{\xi}^{(1)}_{(\nu)} \|_{\rho_3} \leqslant K \| \hat{\xi}_{(\nu)} \|_{\rho_4} \leqslant K \| \hat{\xi} \|_\rho$.

Now we also have $\hat{\xi}^{(1)}_{(\nu)} \mid X_0 = 0$. It follows that

$$\| \hat{\xi}^{(1)}_{(\nu)} \|_\rho \leqslant \gamma/\gamma' \, \| \hat{\xi}^{(1)}_{(\nu)} \|_{\rho_3} \leqslant \gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho \, .$$

Finally we put in $\hat{\mathfrak{U}}^*_9(\rho)$:

$$\hat{\xi}^{(1)} = \Sigma \hat{\xi}^{(1)}_{(\nu)} (t/\rho)^\nu =$$

$$= \Sigma \hat{\xi}_{(\nu)} (t/\rho)^\nu - \Sigma \hat{\eta}_\nu (t/\rho)^\nu - \Sigma a_{\nu\lambda} (t/\rho)^\nu \hat{\mathfrak{b}}_\lambda - \delta(\Sigma \hat{\gamma}_\nu (t/\rho)^\nu)$$

$$= \hat{\xi} - \hat{\eta} - \Sigma a_\lambda \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma} \, .$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of $\hat{\xi}^{(1)}_{(\nu)}$ by (t/ρ) is smaller than $\gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho$ and that with respect to $\hat{\eta}_\nu$ is smaller than $\gamma''' \cdot K \| \hat{\xi} \|_\rho$ we find: $\| \hat{\xi}^{(1)} \|_\rho \leqslant$ $\leqslant \gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho$ and $\| \hat{\eta} \|_\rho \leqslant \gamma''' \cdot K \| \hat{\xi} \|_\rho$ and $\| a_\lambda \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. We take the restriction to $\hat{\mathfrak{V}}(\rho)$ and now $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{V}}(\rho), \mathbf{F})$ is the desired element. Of course we have to choose ρ_4 and then ρ_2 small enough, for example let $\gamma''' < \varepsilon/2K$ and $\gamma \leqslant \varepsilon\gamma'/2K$.

Main Theorem

There exists ρ_2 and a constant K such that if $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ then we can find $a_1, \ldots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in$ $C^{l-1}(\hat{\mathfrak{V}}(\rho), \mathbf{F})$ such that $\hat{\xi} = \sum a_\lambda \hat{\mathfrak{b}}_\lambda + \delta\hat{\eta}$ on $\hat{\mathfrak{V}}(\rho)$ with $\| \hat{\eta} \|_\rho$ and $\| a_\nu \|_\rho \leqslant$ $\leqslant K \| \hat{\xi} \|_\rho$.

Proof. We have one constant K from the smoothing theorem. Now we find ρ_2 with an ε in the Approximation Lemma such that $\varepsilon \cdot K < 1/2$. We shall use this ρ_2 and prove the theorem here. We are given $\hat{\xi}_0 = \hat{\xi} \in$ $Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$. The Approximation Lemma gives $\tilde{\xi}_1 =$

$= \hat{\xi} - \sum \hat{a}_{1\lambda} \hat{\mathfrak{b}}_\lambda - \delta \hat{\gamma}_1$ on $\mathfrak{V}(\rho)$. Here $\hat{\gamma}_1 \in C^{l-1}(\mathfrak{V}(\rho), \mathbf{F})$ and $\| \tilde{\xi}_1 \|_\rho \leqslant \leqslant \varepsilon \| \hat{\xi} \|_\rho$. Now $\tilde{\xi}_1 \in Z^l(\mathfrak{V}(\rho), \mathbf{F})$. The Smoothing Theorem gives $\hat{\xi}_1 \in \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ and $\hat{\eta}_1 \in C^{l-1}(\hat{\mathfrak{V}}(\rho), \mathbf{F})$ such that $\hat{\xi}_1 = \tilde{\xi}_1 + \delta \hat{\eta}_1$ on $\mathfrak{V}'(\rho)$. Here $\| \eta_1 \|_\rho$ and $\| \hat{\xi}_1 \|_\rho \leqslant K \| \tilde{\xi}_1 \|_\rho < \frac{1}{2} \| \hat{\xi} \|_\rho$. Now we use $\hat{\xi}_1$ instead of $\hat{\xi}_0$ as above and get: $\hat{\xi}_2 = \hat{\xi}_1 + \delta \hat{\eta}_2 - \sum a_{2\lambda} \hat{\mathfrak{b}}_\lambda - \delta \hat{\gamma}_2$. Here $\| \hat{\xi}_2 \|_\rho$ and $\| \hat{\eta}_2 \| < \frac{1}{2} \| \hat{\xi}_1 \|_\rho < (\frac{1}{2})^2 \| \hat{\xi} \|_\rho$ and $\| a_{2\lambda} \|_\rho$ and $\| \hat{\gamma}_2 \|_\rho \leqslant \frac{K}{2} \| \hat{\xi} \|_\rho$.

Inductively we get: $\hat{\xi}_n = \hat{\xi}_{n-1} - \sum a_{n\lambda} \hat{\mathfrak{b}}_\lambda - \delta \hat{\gamma}_n + \delta \hat{\eta}_n$. Here $\| \hat{\xi}_n \|_\rho < < 2^{-n} \| \hat{\xi} \|_\rho, \| \hat{\eta}_n \|_\rho \leqslant 2^{-n} \| \hat{\xi} \|_\rho$ and $\| a_{n\lambda} \|_\rho$ and $\| \hat{\gamma}_n \|_\rho \leqslant 2^{-n+1} \cdot K \| \hat{\xi} \|_\rho$ for $n = 1, 2, 3, \ldots$ A summation is now possible. We get $0 = \hat{\xi} - \sum_{n,\lambda} a_{n\lambda} \hat{\mathfrak{b}}_\lambda - - \sum \delta \hat{\gamma}_n + \sum \delta \hat{\eta}_n$. We put $a_\lambda = \sum_n a_{n\lambda}$, $\hat{\eta} = \sum (-\hat{\gamma}_n + \hat{\eta}_n)$ and the theorem follows.

For the proof of the coherence the Main Theorem is needed in a weaker and simpler form.

Main Theorem (*): There exists a positive n-tupel $\rho_2 \leqslant \rho_0$ and cross-sections $S_1, \ldots, S_r \in \Gamma(E^n(\rho_2), \psi_{(l)}(\mathbf{F}))$ such that any $S = \psi_{(l)}(\hat{\underline{\xi}}') \in \Gamma(E^n(\rho'), \psi_{(l)}(\mathbf{F}))$ with $\hat{\underline{\xi}}' \in H^l(X(\rho'), \mathbf{F})$ can be written over $E^n(\rho)$ in the form $S = = \sum_1^n a_\lambda S_\lambda$ with $a_1, \ldots, a_r \in I(E^n(\rho))$. Here $\rho \leqslant \rho_2$ and $\rho < \rho' \leqslant \rho_0$.

Proof. Define $S_\lambda = \psi_{(l)}(\mathbf{b}_\lambda \mid X(\rho_2))$. The cross-section S can be written in the form $S = \psi_{(l)}(\hat{\xi}')$ with $\hat{\xi}' \in Z^l(\hat{\mathfrak{U}}'(\rho'), \mathbf{F})$. We put $\hat{\xi} = \hat{\xi}' \mid \mathfrak{U}(\rho)$. Then $\| \hat{\xi} \|_\rho < \infty$ and we have the representation $\hat{\xi} = = \sum a_\lambda \hat{\mathfrak{b}}_\lambda + \delta \hat{\eta}$. For the cohomology classes we get $\hat{\underline{\xi}} = \sum a_\lambda \mathbf{b}_\lambda$ and for the images $S \mid E^n(\rho)$, this gives $S \mid E^n(\rho) = \psi_{(l)}(\hat{\xi}) = \sum a_\lambda S_\lambda$.

The immediate consequence of this form of the Main Theorem is that the stalk of $\psi_{(l)}(\mathbf{F})$ at the origin (and hence at every point of course) is finitely generated. However this is not yet the full coherence of $\psi_{(l)}(\mathbf{F})$. Nevertheless, the Main Theorem above contains all that is essential, and the rest of the proof is not difficult. We refer to [1, pp. 54-58], or to Knorr [2] for details.