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REPRESENTATIONS OF COMPACT GROUPS AND SPHERICAL HARMONICS

by R. R. Coifman and Guido Weiss [1])

To the memory of Jean Karamata [2])

§ 1. Introductory Remarks

Special functions (in particular, spherical functions) associated with compact groups have been introduced by many authors. See, for example, E. Cartan [4], Dieudonné [5], Godement [6], Vilenkin [11], Weyl [12]. The principal motivation of these authors has been to extend classical results. Our purpose, on the other hand, is to show how these classical results can be obtained in simple and elegant ways by making use of the basic tools of the theory of representations of compact groups. In the usual treatments of the properties of special functions that we derive (see, for example, Bateman et al. [1]) much use is made of the theory of functions and other analytical tools. We do not use the theory of functions at all. For that matter, very little else in analysis is used, and, given the few basic facts of the theory of representations of compact groups listed below, our development is of an elementary algebraic nature. We refer the reader to the fourth chapter of Stein and Weiss [10] for a development of some of these classical results that exploits, in a somewhat different way, the action of the rotation group $SO(n)$ on n-dimensional Euclidean space $\mathbf{R}^n$.

This article is of an expository nature. Probably, few of the results obtained are new. Moreover, some of the methods that we use are known. On the other hand, this treatment of spherical harmonics is not readily available. Yet, it is not solely because of this last mentioned fact that we feel this article should be published; three other reasons motivated our efforts. First, the theory developed is especially elegant. Secondly, many seemingly unrelated topics are brought together. For example, two

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different inner products that are often used in the space of spherical harmonics of degree k (to be defined later) are shown to be constant multiples of each other (this is relation (3.17) of § 3). Perhaps, in this sense, we introduce new material. Thirdly, this development can be of use as a guide to those who want to study more abstract problems in the theory of compact groups.

We would like to take this opportunity to thank Mrs. Mei Chen and Mr. Edward Wilson who have read the manuscript and made several useful suggestions.

Aside from the standard theorems in measure theory, we shall assume that the reader is familiar with those facts that are usually associated with the Peter-Weyl theorem. More precisely, we shall state without proof theorems (1.1) and (1.3) below. We refer the reader to Pontriagin [7] or Pukanszky [8] for these proofs.

Suppose G is a compact group. A *representation* of G is a continous map, $u \to T(u)$, of G into the class of unitary or orthogonal [1]) operators (depending on whether we are dealing with the complex or real case) on a Hilbert space H that satisfies the relation $T(uv) = T(u)\,T(v)$ for all u and v in G. We shall sometimes write T_u instead of $T(u)$.

$L^2(G)$ denotes the space of all complex valued functions f on G satisfying

$$\int_G |f(u)|^2\,du < \infty ,$$

where du is the element of Haar measure on G, which we assume to be so normalized that $\int_G du = 1$. We adopt the usual convention of also letting the symbol $L^2(G)$ denote the Hilbert space of all equivalence classes of square integrable function on G, where two such functions are said to be equivalent provided they are equal almost everywhere. When $H = L^2(G)$ the mapping $u \to R_u$, where $[R_u f](v) = f(u^{-1} v)$ is easily seen to be a representation of G; it is called the *(left) regular representation* of G. The function f_u whose value at v is $f(u^{-1} v)$ is called the (left) *translate of* f by u. Thus, $R_u f = f_u$ for $f \in L^2(G)$ and u in G.

If the representation T acts on the Hilbert space H, a subspace $M \subset H$ is said to be *invariant* under the action of T if $T_u s \in M$ for all $u \in G$ whenever s belongs to M. It follows immediately from the facts that T_u is unitary and that the adjoint, T_u^*, of T_u is $T_{u^{-1}}$, that $M^\perp$, the orthogonal

1) In the usual definition of the notion of a representation the operators are merely assumed to be bounded and invertible. We have defined what is called a *unitary orthogonal representation* of G. Since we shall consider only such representations, our definition avoids the continuous repetition of the words " unitary " and " orthogonal ".

complement of M, is invariant whenever M is invariant. If $\{0\}$ and H are the only invariant subspaces, then the representation T is said to be *irreducible.* A basic result in the theory of representations of compact groups is

THEOREM (1.1). *If the representation* T, *acting on the Hilbert space* H, *is irreducible then* H *is finite dimensional.*

Suppose $\{e_1, e_2, ..., e_d\}$ is an ortho-normal basis of the Hilbert space H of dimension d and L a linear transformation of H into itself. The matrix $A = (a_{ij})$ of L with respect to this basis is defined by the equations

$$Le_i = \sum_{j=1}^{d} a_{ji} e_j, \qquad i = 1, 2, ..., d;$$

thus, the i^{th} column of A consists of the coefficients needed to express Le_i in terms of the basis $\{e_1, e_2 ..., e_d\}$. $A^* = (\overline{a_{ji}})$ denotes the adjoint matrix (the matrix of the adjoint transformation, L^*, defined by the relation $(Ls, t) = (s, L^*t)$ [1]) for all s, t in H).

Thus, if L is unitary $AA^* = I = A^*A$, where I is the identity matrix. $A' = (a_{ji})$ denotes the transpose of $A = (a_{ij})$ (in the real case $A' = A^*$). Finally, $tr\, A$ is the trace of A; that is, $tr\, A = \sum_{j=1}^{d} a_{jj}$.

If T is an irreducible representation acting on H, we can choose an orthonormal basis of H, which must be finite by (1.1), and express T as a unitary matrix (t_{ij}) with respect to this basis. In order to avoid using too much notation we will let the symbol T represent the matrix (t_{ij}) as well. The mapping $u \to T(u) = (t_{ij}(u))$ will then be called a (unitary) matrix valued representation and the fact that multiplication is preserved under this mapping can be expressed by the formula

$$(1.2) \qquad t_{ij}(uv) = \sum_{l=1}^{d} t_{il}(u)\, t_{lj}(v)$$

for all $u, v \in G$. More generally, a *matrix valued representation* is a continuous mapping that assigns to each $u \in G$ a unitary $d \times d$ matrix $T(u) = (t_{ij}(u))$ in such a way that (1.2) is satisfied. If $\mathbf{C}$ denotes the complex number system and $\mathbf{C}^d$ denotes the d-dimensional complex Euclidean space $\{z = (z_1, z_2, ..., z_d): z_j \in \mathbf{C}, j = 1, 2, ..., d\}$ with the usual inner product $z \cdot w = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_d\overline{w_d}$, we also consider $T(u)$

[1]) Unless otherwise stated, the symbol (s, t) denotes the inner product of s and t.

as the unitary operator mapping $z \in \mathbf{C}^d$ into $w = (w_1, w_2, ..., w_d)$, where $w_j = \sum_{l=1}^{d} t_{jl} z_l$ for $j = 1, 2, ..., d$ (that is, if we regard z and w as column vectors, w is the matrix product $T(u) z$). It then follows from (1.2) that $u \to T(u)$ is a representation of G acting on $H = \mathbf{C}^d$ (In the real case we replace $\mathbf{C}$ by $\mathbf{R}$, the real number system, and $\mathbf{C}^d$ by the real Euclidean space $\mathbf{R}^d$).

Suppose T is a matrix valued representation and H_j is the (finite dimensional) subspace of $L^2(G)$ spanned by the entries of the j^{th} column of T. It is an immediate consequence of (1.2) that H_j is invariant under the action of the left regular representation of G.

Two representations S and T, acting on the Hilbert spaces H and K, are said to be *equivalent* when there exists an invertible linear transformation L mapping H onto K such that $T_u L = L S_u$ for all u in G (equivalently, $L^{-1} T_u L = S_u$ for all u in G). A system $\{T^\alpha\}$, $\alpha \in \mathscr{A}$, of irreducible representations of G is said to be *complete* if, given any irreducible representation T, there exists a unique index α such that T and T^α are equivalent. Theorem (1.1), together with the following one, constitute a formulation of the Peter-Weyl theorem:

THEOREM (1.3). *If* $\{\mathrm{T}^\alpha\} = \{(\mathrm{t}^\alpha_{ij})\}$, $\alpha \in \mathscr{A}$, *is a complete system of irreducible matrix valued representations of the compact group* G, *then the collection of functions* $\sqrt{\mathrm{d}_\alpha}\, \mathrm{t}^\alpha_{ij}$ *is an orthonormal basis of* $\mathrm{L}^2(\mathrm{G})$, *where* d_α *is the dimension of the space* H^α *on which* T^α *acts.*

If T is a representation of G then the function mapping $u \in G$ into $tr\{T(u)\} = \chi(u)$ is called the *character of* T. It is clear that if T_1 and T_2 are equivalent representations then the characters of T_1 and T_2 are equal; that is, the character depends only on the equivalence class determined by a representation of G. It is also clear from the orthogonality relations that the character determines the equivalence class of a representation.

COROLLARY (1.4). *Suppose* $\{\mathrm{T}^\alpha\} = \{(\mathrm{t}^\alpha_{ij})\}$, $\alpha \in \mathscr{A}$, *is a complete system of irreducible matrix valued representations of the compact group* G, f *belongs to* $\mathrm{L}^2(\mathrm{G})$ *and* χ^α *denotes the character of* T^α, *then the series*

$$\sum_{\alpha \varepsilon \mathscr{A}} d_\alpha \int_G f(u)\, \overline{\chi^\alpha(uv^{-1})}\, du = \sum_{\alpha \varepsilon \mathscr{A}} d_\alpha \int_G f(u)\, \chi^\alpha(vu^{-1})\, du$$

converges to f(v) *in the* L^2 *norm* [1]).

1) It follows from elementary Hilbert space theory that only a countable number of the summands can be non-zero and the order in which they are taken does not affect the L^2 convergence of the above series.

Proof. By theorem (1.3), the functions $\sqrt{d_\alpha}\, t^\alpha_{ij}$ form an orthonormal basis of $L^2(G)$. Thus,

$$f = \sum_{\alpha \varepsilon \mathscr{A}} \left(\sum_{i,j=1}^{d_\alpha} c^\alpha_{ij}\, t^\alpha_{ij} \right), \tag{1.5}$$

where $c^\alpha_{ij} = d_\alpha \int_G f(u)\, \overline{t^\alpha_{ij}(u)}\, du$ and the convergence is in the L^2 norm.
If C^α is the matrix (c^α_{ij}) and $[T^\alpha(v)]'$ is the transpose of $T^\alpha(v)$, then

$$\sum_{i,j=1}^{d_\alpha} c^\alpha_{ij}\, t^\alpha_{ij}(v) = tr\{ C^\alpha [T^\alpha(v)]' \} = d_\alpha \int_G f(u)\, tr\{ \overline{T^\alpha(u)}\, [T^\alpha(v)]' \}\, du .$$

Since $T^\alpha(v)$ is unitary and its inverse is $T^\alpha(v^{-1})$ we have $[T^\alpha(v)]' = \overline{T^\alpha(v^{-1})}$. Thus,

$$d_\alpha \int_G f(u)\, tr\{ \overline{T^\alpha(u)}\, [T^\alpha(v)]' \}\, du = d_\alpha \int_G f(u)\, tr\{ \overline{T^\alpha(u)\, T^\alpha(v^{-1})} \}\, du$$

$$= d_\alpha \int_G f(u)\, \overline{tr\{ T^\alpha(uv^{-1}) \}}\, du = d_\alpha \int_G f(u)\, tr\{ T^\alpha(vu^{-1})\, du$$

and the corollary is proved.

THEOREM (1.6). *Suppose* $\mathrm{T} = (\mathrm{t}_{ij})$, $1 \leqq \mathrm{i}, \mathrm{j} \leqq \mathrm{d}$, *is an irreducible matrix valued representation of* G *and* $\mathrm{H}_j \subset \mathrm{L}^2(\mathrm{G})$ *is the subspace spanned by the entries* $\mathrm{t}_{1j}, \mathrm{t}_{2j}, \ldots, \mathrm{t}_{dj}$ *of the* j^{th} *column of* T. *Then the restriction,* $\mathrm{R}^{(j)}$, *of the left regular representation of* G *to* H_j *is an irreducible representation of* G. *Moreover,* $\mathrm{R}^{(j)}$ *and* $\mathrm{R}^{(k)}$ *are equivalent for* $1 \leqq \mathrm{j}, \mathrm{k} \leqq \mathrm{d}$ *and each of these representations is equivalent to the representation* $\overline{T}$ *on* H *whose value at* $\mathrm{u} \in \mathrm{G}$ *is* $\overline{T}_u = \mathrm{T}'_{u^{-1}}$.

Proof. We have already observed that (1.2) implied that H_j is invariant under the action of the left regular representation. To show that $R^{(j)}$ is irreducible we consider the standard orthonormal basis $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_d = (0, 0, \ldots, 1)$ of $H = \mathbf{C}^d$ and define a linear transformation, L, on H into H_j by putting $Le_i = \sqrt{d}\, t_{ij}$, $1 \leqq i \leqq d$. From the definition we see that $\overline{T}$ is the matrix valued representation having coefficients that are complex conjugates of the ones ocurring in T. By (1.2) we then have

$$(L\overline{T}_u e_i)(v) = L\left(\sum_{l=1}^{d} \overline{t_{li}(u)}\, e_l \right)(v) = \sqrt{d} \sum_{l=1}^{d} \overline{t_{li}(u)}\, t_{lj}(v)$$

$$= \sqrt{d} \sum_{l=1}^{d} t_{il}(u^{-1})\, t_{lj}(v) = \sqrt{d}\, t_{ij}(u^{-1}v) = (R^{(j)}_u\, Le_i)(v)$$

for all $u, v \in G$ and $i = 1, 2, ..., d$. Thus, $L\overline{T} = R^{(j)} L$ which shows that each of the representations $R^{(j)}$ are equivalent to $\overline{T}$. The theorem now follows immediately.[1])

§ 2. THE CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF SOME SPECIAL GROUPS

In this section we show how one can obtain irreducible representations of some of the classical compact groups. In many cases we describe several representations that are equivalent to each other. We shall see that often one of the members of this equivalence class of representations has special features that make the study of certain properties particularly easy.

If we are given two finite dimensional representations of a compact group G that act on the Hilbert spaces H and K, we can obtain a third representation of G by constructing the *tensor product of* H *and* K. The classical definition of this concept is the following: We choose orthonormal bases $\{ e_1, e_2, ..., e_m \}$ and $\{ f_1, f_2, ..., f_n \}$ of H and K, respectively, and we assign to each of the $m \cdot n$ pairs (e_i, f_j) a " product " $e_i \otimes f_j$, called the *tensor product of the elements* e_i and f_j. We then obtain a new Hilbert space by considering all the linear combinations

$$\sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) ,$$

defining addition and scalar multiplication by letting

$$\sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) + \sum_{i,j=1}^{m,n} b_{ij} (e_i \otimes f_j) = \sum_{i,j=1}^{m,n} (a_{ij} + b_{ij}) (e_i \otimes f_j) ,$$

$$c \sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) = \sum_{i,j=1}^{m,n} c a_{ij} (e_i \otimes f_j) ,$$

and the inner product by letting

$$\left(\sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) , \sum_{i,j=1}^{m,n} b_{ij} (e_i \otimes f_j) \right) = \sum_{i,j=1}^{m,n} a_{ij} \overline{b_{ij}} .$$

This space is denoted by $H \otimes K$ and is called the *tensor product of* H *and* K. It is clear that $\{ e_i \otimes f_j \}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, is an orthonormal basis

1) We observe that L is an isometry. We will make use of this fact later in § 3.