§2. The construction of irreducible representations of some special groups

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for all $u, v \in G$ and $i = 1, 2, \ldots, d$. Thus, $L\overline{T} = R^{(j)} L$ which shows that each of the representations $R^{(j)}$ are equivalent to $\overline{T}$. The theorem now follows immediately.[1])

§ 2. THE CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF SOME SPECIAL GROUPS

In this section we show how one can obtain irreducible representations of some of the classical compact groups. In many cases we describe several representations that are equivalent to each other. We shall see that often one of the members of this equivalence class of representations has special features that make the study of certain properties particularly easy.

If we are given two finite dimensional representations of a compact group G that act on the Hilbert spaces H and K, we can obtain a third representation of G by constructing the *tensor product of* H *and* K. The classical definition of this concept is the following: We choose orthonormal bases $\{ e_1, e_2, \ldots, e_m \}$ and $\{ f_1, f_2, \ldots, f_n \}$ of H and K, respectively, and we assign to each of the $m \cdot n$ pairs (e_i, f_j) a " product " $e_i \otimes f_j$, called the *tensor product of the elements* e_i and f_j. We then obtain a new Hilbert space by considering all the linear combinations

$$\sum_{i,j=1}^{m,n} a_{ij}(e_i \otimes f_j),$$

defining addition and scalar multiplication by letting

$$\sum_{i,j=1}^{m,n} a_{ij}(e_i \otimes f_j) + \sum_{i,j=1}^{m,n} b_{ij}(e_i \otimes f_j) = \sum_{i,j=1}^{m,n} (a_{ij} + b_{ij})(e_i \otimes f_j),$$

$$c \sum_{i,j=1}^{m,n} a_{ij}(e_i \otimes f_j) = \sum_{i,j=1}^{m,n} c a_{ij}(e_i \otimes f_j),$$

and the inner product by letting

$$\left(\sum_{i,j=1}^{m,n} a_{ij}(e_i \otimes f_j), \sum_{i,j=1}^{m,n} b_{ij}(e_i \otimes f_j) \right) = \sum_{i,j=1}^{m,n} a_{ij} \overline{b_{ij}}.$$

This space is denoted by $H \otimes K$ and is called the *tensor product of* H *and* K. It is clear that $\{ e_i \otimes f_j \}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, is an orthonormal basis

[1]) We observe that L is an isometry. We will make use of this fact later in § 3.

of $H \otimes K$. If a $= \sum_{i=1}^{m} a_i e_i \in H$ and $b = \sum_{j=1}^{n} b_j f_j \in K$ the tensor product of the elements a and b is defined to be the element $a \otimes b = \sum_{i,j=1}^{m,n} a_i b_j (e_i \otimes f_j)$ of $H \otimes K$.

$H \otimes K$ can be identified with the linear space $\mathscr{L}(H, K)$ of all linear transformations mapping H into K in the following way: to each element $e_i \otimes f_j$ we assign the transformation mapping e_i onto f_j, and e_k, for $k \neq i$, onto the zero vector of K. We then extend this correspondence linearly to all of $H \otimes K$. If we represent the elements of $\mathscr{L}(H, K)$ as $n \times m$ matrices with respect to the two bases in question, this correspondence assigns the matrix $A = (a_{ji})$ to the element $\sum_{i,j=1}^{m,n} a_{ji} (e_i \otimes f_j)$. If $B = (b_{ji})$ is another such matrix, it is easy to check that the inner product of the elements of $H \otimes K$ corresponding to A and B is $tr\,(AB^*)$. We identify $H \otimes K$ with $\mathscr{L}(H, K)$; moreover, we will not use different notation to distinguish the latter space from the corresponding linear space of $m \times n$ matrices.

If $u \in \mathscr{L}(H, H)$ and $v \in \mathscr{L}(K, K)$, we obtain a linear transformation $u \otimes v$ of $H \otimes K$ into itself by letting

$$(u \otimes v)\, t = vtu' \tag{2.1}$$

for all $t \in H \otimes K$ (we are regarding t as a member of $\mathscr{L}(H, K)$ and u' is the transformation whose matrix with respect to $\{ e_1, e_2, \ldots, e_m \}$ is the transpose of the matrix of u). The transformation $u \otimes v$ is called the *tensor product* of u and v. An equivalent way of defining this tensor product is the following one: Suppose

$$ue_i = \sum_{l=1}^{m} a_{li}\, e_l \text{ and } vf_j = \sum_{k=1}^{n} b_{kj} f_k \text{ then we let}$$

$$(u \otimes v)(e_i \otimes f_j) = (ue_i) \otimes (vf_j) = \sum_{l,k=1}^{m,n} a_{li}\, b_{kj}\, (e_l \otimes f_k) \tag{2.1'}$$

and extend $u \otimes v$ linearly over all of $H \otimes K$.

In order to see that (2.1) and (2.1′) define the same transformation, it clearly suffices to show that they agree when applied to the basis vectors $t_{ij} = e_i \otimes f_j$. From (2.1) we have $(u \otimes v)\, t_{ij} = vt_{ij}\, u'$. Thus,

$$[(u \otimes v)\, t_{ij}]\, e_r = vt_{ij} \sum_{k=1}^{m} a_{rk}\, e_k = va_{ri} f_j = a_{ri} \sum_{k=1}^{n} b_{kj} f_k\,.$$

On the other hand, from (2.1′) we have

$$[(u \otimes v)\, t_{ij}]\, e_r = \sum_{l,k=1}^{m,n} a_{li}\, b_{kj}\, t_{lk}\, e_r = \sum_{k=1}^{m} a_{ri}\, b_{kj}\, f_k\,.$$

Thus, in either case we obtain the same transformation.

We now show that if u and v are unitary so is $u \otimes v$. Since $u \otimes v$ is a linear transformation on a finite dimensional Hilbert space it suffices to prove that it is an isometry. But, if $t \in H \otimes K$ we have

$$\|(u \otimes v)\, t\|^2 = ((u \otimes v)\, t\,,\ (u \otimes v)\, t) = tr\{vtu'(vtu')^*\}$$

$$= tr\{vt(u^*u)'\, t^*\, v^*\} = tr\{vtt^*\, v^*\} = tr\{tt^*\} = (t, t) = \|t\|^2\,.$$

If $u \to S_u$ is a representation of G acting on H and $u \to T_u$ is a representation of G acting on K, then

$$(S_{uv} \otimes T_{uv})\, t = T_{uv}\, tS'_{uv} = T_u\, T_v\, tS'_v\, S'_u = (S_u \otimes T_u)\,(S_v \otimes T_v)\, t$$

for all $u, v \in G$ and $t \in H \otimes K$.

We can summarize these results in the following way:

THEOREM (2.2). *If* $\mathrm{u} \to \mathrm{S_u}$ *and* $\mathrm{u} \to \mathrm{T_u}$ *are two representations of* G *acting on the Hilbert space* H *and* K *respectively, then the mapping* $\mathrm{u} \to \mathrm{S_u} \otimes \mathrm{T_u}$ *is a representation of* G *acting on the tensor product* $\mathrm{H} \otimes \mathrm{K}$.

If $H_1, H_2, \ldots, H_k$ are finite dimensional Hilbert spaces we define their *tensor product* $\bigotimes_{j=1}^{k} H_j$ inductively by letting

$$\bigotimes_{j=1}^{k} H_j = \left(\bigotimes_{j=1}^{k-1} H_j\right) \otimes H_k$$

for $k > 2$. We ahsll often write $H_1 \otimes H_2 \otimes \ldots \otimes H_k$ instead of $\bigotimes_{j=1}^{k} H_j$. By making obvious identifications we may regard this product to be associative; the same remark applies to the k-fold tensor products $a_1 \otimes a_2 \otimes \ldots \otimes a_k$, where $a_j \in H_j$ for $1 \leqq j \leqq k$. We shall be interested mostly in the case $H_1 = H_2 = \ldots = H_k = H$ and we shall denote the tensor product of k copies of H by $\mathscr{T}^{(k)}(H)$ or, if there is no chance of confusion, simply by $\mathscr{T}^{(k)}$. We shall fix k for the remainder of this discussion.

If $\{e_1\, e_2, \ldots, e_n\}$ is an orthonormal basis of H and Δ is the set of all k-tuples of integers, $m = (m_1, m_2, \ldots, m_k)$, with $1 \leqq m_1, m_2, \ldots, m_k \leqq n$, then the collection $\{ \varepsilon_m \}_{m \varepsilon \Delta}$, where $\varepsilon_m = e_{m_1} \otimes \dots \otimes e_{m_k}$, is an oitho-normal basis of $\mathcal{T}^{(k)}$. Thus, the general element t of this tensor product has the representation

$$t = \sum_{m \varepsilon \Delta} t_m \, \varepsilon_m ,$$

where the t_m's are complex numbers.

The tensor product $u_1 \otimes u_2 \otimes \dots \otimes u_k$ of k linear transformations $u_1, u_2, \dots, u_k$ mapping H into itself can also be defined inductively by extending (2.1′). Its action on the basis elements ε_m is given by

$$(u_1 \otimes u_2 \otimes \dots \otimes u_k) \, \varepsilon_m = (u_1 \, e_{m_1}) \otimes (u_2 \, e_{m_2}) \otimes \dots \otimes (u_k \, e_{m_k}) .$$

When $u_1 = u_2 = \dots = u_k = u$ we denote this tensor product by T_u. If

$$\begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix}$$

is the matrix of u with respect to the basis $\{ e_1, e_2, \dots, e_n \}$ we then have

$$T_u \, \varepsilon_m = \sum_{j \varepsilon \Delta} (T_u)_{j,m} \cdot \varepsilon_j , \tag{2.3}$$

where

$$(T_u)_{j,m} = u_{j_1 m_1} u_{j_2 m_2} \dots u_{j_k m_k}$$

for $j = (j_1, j_2, \dots, j_k)$ and $m = (m_1, m_2, \dots m_k)$ in Δ. It follows from theorem (2.2) that the mapping $u \to T_u$ is a representation of the unitary group of transformations on H. This representation acts on $\mathcal{T}^{(k)}$. When $k > 1$ this is not an irreducible representation. In order to exhibit a proper invariant subspace of $\mathcal{T}^{(k)}$ we introduce the subspace $\mathcal{S}^{(k)}$ of *symmetric tensors of degree* k: If τ is a permutation of $\{ 1, 2, \dots, k \}$ and $m \in \Delta$ we let $\tau m = \{ m_{\tau(1)}, m_{\tau(2)}, \dots, m_{\tau(k)} \}$. Then

$$\mathcal{S}^{(k)} = \{ t = \sum_{m \varepsilon \Delta} t_m \, \varepsilon_m \quad \text{in} \quad \mathcal{T}^{(k)} : t_{\tau m} = t_m$$

for all permutations τ and $m \in \Delta$ }.

THEOREM (2.4). *The subspace* $\mathcal{S}^{(k)}$ *is invariant under the action of the representation* $\mathrm{u} \to \mathrm{T}_u = \mathrm{u} \otimes \mathrm{u} \otimes \dots \otimes \mathrm{u}$ *of the unitary group of transformations on* H.

Proof. We first observe that for any permutation τ of $\{1, 2, ..., k\}$ we have

$$(2.5) \qquad (T_u)_{\tau j, m} = (T_u)_{j, \tau^{-1} m} .$$

This equality is an immediate consequence of the definition of the coefficients $(T_u)_{j,m}$ (see (2.3)) when τ is a transposition. The general case is then obtained by writing τ as a product of transpositions.

Consider the set of all n tuples $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of non-negative integers satisfying $\|\alpha\| = \alpha_1 + \alpha_2 + ... + \alpha_n = k$ and let Δ_α be the set of all the $m = (m_1, m_2, ..., m_k)$ in Δ such that i, $1 \leqq i \leqq n$, is one of the components of m precisely α_i times. We then have $\Delta = \bigcup_{\|\alpha\|=k} \Delta_\alpha$ and if $m \in \Delta_\alpha$ then τm also belongs to Δ_α. Moreover, it is easy to see that the collection of all, $\sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} \varepsilon_m$, $\|\alpha\| = k$, is a basis for $\mathscr{S}^{(k)}$. Consequently, it suffices to show that $T_u \sigma_\alpha \in \mathscr{S}^{(k)}$ when $\|\alpha\| = k$. By (2.3) we have

$$T_u \sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} T_u \varepsilon_m = \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j \varepsilon \Delta} (T_u)_{j,m} \varepsilon_j \Big) = \sum_{j \varepsilon \Delta} \Big(\sum_{m \varepsilon \Delta_\alpha} (T_u)_{j,m} \Big) \varepsilon_j .$$

If τ is any permutation, it follows from (2.5) that

$$\sum_{m \varepsilon \Delta_\alpha} (T_u)_{\tau j, m} = \sum_{m \varepsilon \Delta_\alpha} (T_u)_{j, \tau^{-1} m} = \sum_{m \varepsilon \Delta_\alpha} (T_u)_{j,m} .$$

Thus, the coefficient of ε_j equals that of $\varepsilon_{\tau j}$ in the above expansion of $T_u \sigma_\alpha$. Hence, $T_u \sigma_\alpha \in \mathscr{S}^{(k)}$ and the theorem is proved.

We shall show that the restriction of this representation $u \to T_u$ to $\mathscr{S}^{(k)}$ is irreducible. This is particularly simple to do if we examine a representation that is equivalent to it that acts on the vector space $\mathscr{P}^{(k)} = \mathscr{P}^{(k,n)}$ of homogeneous polynomial functions of degree k of the n complex variables $z = (z_1, z_2, ..., z_n)$. We use the following notation in our discussion of this space: If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is an n-tuple of non-negative integers we put $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$ when $z = (z_1, z_2, ..., z_n) \in \mathbf{C}^n$, $\alpha! = \alpha_1! \alpha_2! ... \alpha_n!$ and, when $\|\alpha\| = k$, $\binom{k}{\alpha} = k!/\alpha!$ (note that $\binom{k}{\alpha}$ is the number of elements in the set Δ_α we introduced in the last proof). The polynomials

$$p_\alpha(z) = \binom{k}{\alpha} z^\alpha = \frac{k!}{\alpha_1! \alpha_2! ... \alpha_n!} z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n},$$

$\|\alpha\| = k$, form a basis of $\mathscr{P}^{(k)}$. We have just observed, however, that the elements $\sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} \varepsilon_m$, $\|\alpha\| = k$, form a basis of the space $\mathscr{S}^{(k)}$ of symmetric tensors of degree k. We can, therefore, extend the map $\sigma_\alpha \to p_\alpha$ linearly and obtain a one to one linear transformation $\pi = \pi^{(k)}$ of $\mathscr{S}^{(k)}$ onto $\mathscr{P}^{(k)}$. This transformation is an isometry if we introduce an inner product on $\mathscr{P}^{(k)}$ by letting $(\pi s, \pi t) = (s, t)$ for all symmetric tensors s and t in $\mathscr{S}^{(k)}$. Obvious consequences of these definitions are: if $p = \sum_{\|\alpha\|=k} c_\alpha p_\alpha$ and $q = \sum_{\|\alpha\|=k} d_\alpha p_\alpha$ then

$$(p, q) = \sum_{\|\alpha\|=k} c_\alpha \bar{d}_\alpha (\sigma_\alpha, \sigma_\alpha) = \sum_{\|\alpha\|=k} c_\alpha \bar{d}_\alpha \binom{k}{\alpha}. \tag{2.6}$$

On the other hand, if $p(z) = \sum_{\|\alpha\|=k} a_\alpha z^\alpha$ and $q(z) = \sum_{\|\alpha\|=k} b_\alpha z^\alpha$ then

$$(p, q) = \sum_{\|\alpha\|=k} \frac{a_\alpha \bar{b}_\alpha}{\binom{k}{\alpha}}. \tag{2.6'}$$

Let $D = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right)$, $D^\alpha = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$

and, for $p(z) = \sum_{\|\alpha\|=k} a_\alpha z^\alpha$ in $\mathscr{P}^{(k)}$, put

$$p(D) = \sum_{\|\alpha\|=k} a_\alpha D^\alpha .$$

Then, if $q(z) = \sum_{\|\alpha\|=k} b_\alpha z^\alpha$ we have

$$(p, q) = \frac{1}{k!} \sum_{\|\alpha\|=k} \alpha!\, a_\alpha \bar{b}_\alpha = \frac{1}{k!} p(D)\bar{q}, \tag{2.6''}$$

where

$$\bar{q}(z) = \sum_{\|\alpha\|=k} \bar{b}_\alpha z^\alpha .$$ [1]

Theorem (2.7). *For each unitary transformation* u *on* H *let* S_u *be the transformation on* $\mathscr{P}^{(k)}$ *that maps a polynomial function* p *into the polynomial function* q (z) = p (u′ z), *where* u′ *is the transpose of the matrix of* u *with respect to the orthonormal basis* $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$. *Then* $\mathrm{S}: \mathrm{u} \to \mathrm{S}_u$ *is a representation of the unitary group of transformations on* H *that is equivalent to* $\mathrm{T}: \mathrm{u} \to \mathrm{T}_u$. *In fact,*

$$\pi T_u = S_u \pi \tag{2.8}$$

for all unitary transformations u *on* H.

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[1] Il Calderón [2] and in the previously mentioned Chapter IV of Stein and Weiss [10] the inner product on $\mathscr{P}^{(k)}$ was introduced by formula (2.6″). It appears much more natural in this context when we see its connection with the inner product of $\mathscr{S}^{(k)}$.

Proof. Let $L\colon \mathscr{T}^{(k)} \to \mathscr{P}^{(k)}$ be the linear transformation that maps

$$\varepsilon_m = e_{m_1} \otimes e_{m_2} \otimes \dots \otimes e_{m_k} \quad \text{into} \quad z_{m_1} z_{m_2} \dots z_{m_k} .$$

Since Δ_α has $\binom{k}{\alpha}$ elements it follows that

$$L\sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} L\varepsilon_m = \binom{k}{\alpha} z^\alpha = p_\alpha(z) = \pi\sigma_\alpha .$$

That is, π is the restriction of L to $\mathscr{S}^{(k)}$.

In order to show (2.8) it suffices to show that $\pi T_u \sigma_\alpha = S_u \pi\sigma_\alpha$ for all unitary transformations u on H and $\| \alpha \| = k$. We have, by (2.3),

$$T_u \sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} T_u \varepsilon_m = \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j \varepsilon \Delta} (T_u)_{j,m} \varepsilon_j \Big)$$

where

$$(T_u)_{j,m} \varepsilon_j = u_{j_1 m_1} u_{j_2 m_2} \dots u_{j_k m_k} \varepsilon_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_k} .$$

Thus,

$$LT_u \sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j_1, \dots, j_n = 1}^{n} u_{j_1 m_1} \dots u_{j_k m_k} z_{j_1} \dots z_{j_k} \Big)$$

$$= \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j=1}^{n} u_{j m_1} z_j \Big) \Big(\sum_{j=1}^{n} u_{j m_2} z_j \Big) \dots \Big(\sum_{j=1}^{n} u_{j m_k} z_j \Big) = p_\alpha(u' z) = S_u \pi\sigma_\alpha .$$

Since $T_u \sigma_\alpha \in \mathscr{S}^{(k)}$ by (2.4), we have $LT_u\sigma_\alpha = \pi T_u \sigma_\alpha$. Hence, $\pi T_u \sigma_\alpha = S_u\pi\sigma_\alpha$ for $\| \alpha \| = k$. This shows that (2.8) is true. The fact that $S_{u_1 u_2} = S_{u_1} S_{u_2}$ for any two unitary transformations u_1 and u_2 is immediate. In order to establish the theorem, therefore, we must show that S_u is unitary. But, if p and q belong to $\mathscr{P}^{(k)}$ there exist (unique) symmetric tensors s and t such that $p = \pi s$ and $q = \pi t$. Then,

$$(S_u p, S_u q) = (S_u \pi s, S_u \pi t) = (\pi T_u s, \pi T_u t) = (T_u s, T_u t)$$

$$= (s, t) = (\pi s, \pi t) = (p, q) ,$$

which shows that S_u is unitary.

THEOREM (2.9). *The representation* $\mathrm{S}\colon \mathrm{u} \to \mathrm{S}_u$ *is irreducible.*

Proof. We first observe that it suffices to show that any linear transformation A on $\mathscr{P}^{(k)}$ such that $AS_u = S_u A$ for all unitary u must be a constant times the identity. To see that this is the case, suppose S leaves a subspace $V \subset \mathscr{P}^{(k)}$ invariant and P is the projection of $\mathscr{P}^{(k)}$ onto V. Since V is also invariant it follows that $PS_u = S_u P$ for all unitary transformations u on H. Consequently, P must be a constant times the identity transformation on $\mathscr{P}^{(k)}$. But, since P is a projection, this constant must be either 0 or 1; thus, V is either $\{ 0 \}$ or $\mathscr{P}^{(k)}$, which means that S is irreducible.

Suppose, then, that the operator A commutes with the representation S and let u be the unitary operator whose matrix with respect to $\{ e_1, e_2, ..., e_n \}$ is diagonal with $u_{jj} = e^{i\theta_j}$, $1 \leqq j \leqq n$. Then

$$(S_u p_\alpha)(z) = p_\alpha (u' z) = \binom{k}{\alpha} (e^{i\theta_1} z_1)^{\alpha_1} (e^{i\theta_2} z_2)^{\alpha_2} \dots (e^{i\theta_n} z_n)^{\alpha_n} .$$

If we let $\theta = (\theta_1, \theta_2, ..., \theta_n)$ and $\theta . \alpha = \theta_1 \alpha_1 + \theta_2 \alpha_2 + ... + \theta_n \alpha_n$ we can express the action of S_u by the simple formula

$$S_u p_\alpha = e^{i\theta \cdot \alpha} p_\alpha .$$

Suppose A, on the other hand, transforms the basis elements p_α in the following manner

$$A p_\alpha = \sum_{\| \beta \| = k} a_{\beta\alpha} p_\beta .$$

Since $AS_u = S_u A$ we then must have

$$\sum_{\| \beta \| = k} a_{\beta\alpha} e^{i\theta \cdot \beta} p_\beta = S_u A p_\alpha = AS_u p_\alpha = \sum_{\| \beta \| = k} a_{\beta\alpha} e^{i\theta \cdot \alpha} p_\beta .$$

Consequently, $a_{\beta\alpha} e^{i\theta \cdot \beta} = a_{\beta\alpha} e^{i\theta \cdot \alpha}$ for all n-tuples $\theta = (\theta_1, \theta_2, \theta_3, ..., \theta_n)$. Thus, either, $\alpha = \beta$ or $a_{\beta\alpha} = 0$. It follows that $AP_\alpha = a_{\alpha\alpha} p_\alpha$ for $\| \alpha \| = k$.

If $BA = AB$, where B is a linear transformation satisfying

$$B p_\alpha = \sum_{\| \beta \| = k} b_{\beta\alpha} P_\beta \quad \text{for} \quad \| a \| = k ,$$

we must have

$$\sum_{\| \beta \| = k} b_{\beta\alpha} a_{\alpha\alpha} p_\beta = BA p_\alpha = AB p_\alpha = \sum_{\| \beta \| = k} b_{\beta\alpha} a_{\beta\beta} p_\beta .$$

Thus, $a_{\alpha\alpha} b_{\beta\alpha} = b_{\beta\alpha} a_{\beta\beta}$. Thus, if we can find such an operator B with $b_{\beta\alpha} \neq 0$ for some α and all β ($\| \beta \| = k$) it would follow that $a_{\alpha\alpha} = a_{\beta\beta}$ for all α and β. This would show that A is a constant times the identity operator and the theorem would be proved. In order to obtain such a B we choose a unitary operator u on H whose matrix with respect to $\{ e_1, e_1, ..., e_n \}$ has no zero elements in the first column (i.e. u_{j1},

$j = 1, 2, \ldots, n$, is not zero). With $\alpha = (k, 0, \ldots, 0)$ (that is, $p_\alpha(z) = z_1^k$) we then have by (2.7)

$$(S_u p_\alpha)(z) = p_\alpha(u' z) = (z_1 u_{11} + z_2 u_{21} + \ldots + z_n u_{n1})^k$$
$$= \sum_{\|\beta\| = k} u_{11}^{\beta_1} u_{21}^{\beta_2} \ldots u_{n1}^{\beta_n} z^\beta .$$

Clearly,

$$b_{\beta\alpha} = u_{11}^{\beta_1} u_{21}^{\beta_2} \ldots u_{n1}^{\beta_n} \neq 0$$

for all β satisfying $\|\beta\| = k$ and the theorem is proved.

Since S and T are equivalent representations (theorem (2.7)) we have the following corollary of (2.9):

COROLLARY (2.10). *The representation* T: u→T_u *is irreducible.*

When the Hilbert space H is n-dimensional Euclidean space $\mathbf{C}^n$ the group of all unitary operators on H is called the *unitary group on* $\mathbf{C}^n$ and is denoted by $U(n)$. The same notation will be used for the group of matrices of the operators in $U(n)$ with respect to the standard basis $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots e_n = (0, 0, \ldots, 1)$. The *special unitary group*, $SU(n)$, is the subgroup of those elements of $U(n)$ having determinant 1. The spaces $\mathscr{S}^{(k)}$ and $\mathscr{P}^{(k)}$ are obviously invariant under the restrictions of the representations T and S to $SU(n)$. It is not hard to show that these restrictions are irreducible representations. By the equivalence (2.8) it suffices to show that this is true for the representation S. But this requires only one simple change in the proof of theorem (2.9): Instead of the equality $a_{\beta\alpha}\, e^{i\theta\cdot\alpha} = a_{\beta\alpha}\, e^{i\theta\cdot\beta}$ holding for all n-tuples $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ we obtain this same equality for all n-tuples θ satisfying $\theta_1 + \theta_2 + \ldots + \theta_n = 0$ (thus, det $u = 1$). This suffices for obtaining the conclusion that either $\alpha = \beta$ or $a_{\beta\alpha} = 0$. For, if $a_{\beta\alpha} \neq 0$ then $e^{i\theta\cdot\alpha} = e^{i\theta\cdot\beta}$ for all such n-tuples θ. Thus, if r is any real number we must have $e^{ir\theta\cdot(\alpha-\beta)} = 1$ whenever $\theta_1 + \theta_2 + \ldots + \theta_n = 0$. But $(\alpha_1 - \beta_1) + {} + (\alpha_2 - \beta_2) + \ldots + (\alpha_n - \beta_n) = k - k = 0$. This allows us to choose $\theta = \alpha - \beta$ and we obtain

$$e^{ir(\alpha-\beta)\cdot(\alpha-\beta)} = 1$$

for all real numbers r, which can occur only if $\alpha = \beta$. We have shown, therefore, the following corollary:

COROLLARY (2.11). *The restrictions of* S *and* T *to* SU (n) *are equivalent irreducible representations of* SU (n).

It is clearly not reasonable to expect that the restriction of an irreducible representation of $U(n)$ to a subgroup is also an irreducible representation of the subgroup. If we consider the *orthogonal group* $0(n)$ (i.e. those operators in $U(n)$ whose matrices with respect to $\{e_1, e_2, ..., e_n\}$ have only real entries) and restrict S, or T, to $0(n)$ we do not obtain an irreducible representation. In studying the problem of how the space $\mathscr{P}^{(k)}$ can be decomposed into subspaces that are invariant under the action of S restricted to $0(n)$ it is more natrual to consider the elements of $\mathscr{P}^{(k)}$ to be polynomial functions of n real variables. Thus, if we denote this restriction by S^0 and $x = (x_1, x_2, ..., x_n)$ is a point of n-dimensional real Euclidean space $\mathbf{R}^n$ then $(S_u^0 p)(x) = p(u' x)$ for each $u \in 0(n)$ and $p \in \mathscr{P}^{(k)}$. We denote the inner product of two points $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ of $\mathbf{R}^n$ by $x \cdot y = x_1, y_1, + x_1 y_2 + ... + x_n y_n$; $|x| = \sqrt{x \cdot x}$ is then the Euclidean norm of x. Since this inner product is invariant under the action of $0(n)$ (that is, $(ux) \cdot (uy) = x \cdot y$ whenever $u \in 0(n)$) the subspace

$$|x|^2 \mathscr{P}^{(k-2)} = \{p \in \mathscr{P}^{(k)} : p(x) = |x|^2 q(x) \quad \text{with} \quad q \in \mathscr{P}^{(k-2)}\}$$

when $k > 1$, and

$$|x|^2 \mathscr{P}^{(k-2)} = \{0\} \quad \text{when} \quad k = 0,1 .$$

is invariant under the action of S^0. Consequently, the orthogonal complement $\mathscr{H}_n^{(k)}$ of this subspace is also invariant. We let $S^{k,n}$ denote the restriction of S^0 to $\mathscr{H}_n^{(k)}$. Thus, for each $u \in 0(n)$, $S_u^{k,n} = S^{k,n}(u)$ is the operator mapping a polynomial $p \in \mathscr{H}_n^{(k)}$ into the polynomial $q = S_u^{k,n} p$ whose value at x is $p(u' x) = p(u^{-1} x)$.

We recall that the differential operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

is called the *Laplacian*. If a function f defined in a region of $\mathbf{R}^n$ satisfies $\Delta f = 0$ then f is called a *harmonic function.*

THEOREM (2.12). *The representation* $\mathrm{S}^{k,n}$ *of* 0 (n) *is irreducible. The space* $\mathscr{H}_n^{(k)}$ *on which it acts consists of all the harmonic polynomial functions on* $\mathbf{R}^n$ *that are homogeneous of degree* k. $\mathscr{P}^{(k)}$ *is the orthogonal direct sum of* $\mathscr{H}_n^{(k)}$ *and the subspaces*

$$|x|^{2j} \mathscr{H}_n^{(k-2j)} = \{p \in \mathscr{P}^{(k)}: \ p(x) = |x|^{2j} q(x), \ q \in \mathscr{H}^{(k-2j)}\}, \quad 1 \leqq j \leqq k/2 ;$$

moreover, the restriction of S^0 *to each of these subspaces is an irreducible representation of* 0 (n).

Proof. By (2.6″) and the definition of $\mathscr{H}_n^{(k)}$ we must have, for $p \in \mathscr{H}_n^{(k)}$, $0 = k\,!\,(r, p) = r(D)\,\bar{p}$ for all $r \in |x|^2\,\mathscr{P}^{(k-2)}$, $k \geqq 2$ (when $k < 2$ polynomials in $\mathscr{P}^{(k)}$ are obviously harmonic). In particular, if we choose $r(x) = |x|^2\,q(x)$ where $q = \Delta p$ we have, since $|x|^2\,q(x) = q(x)\,|x|^2$, $0 = q(D)\,\Delta\bar{p} = (\Delta p, \Delta p)$. But this means that $\Delta p = 0$; thus, p is harmonic. The converse, that each harmonic polynomial that is homogeneous of degree k belongs to $\mathscr{H}_n^{(k)}$, is evident.

If we show that $S^{k,n}$ is irreducible the rest of the theorem follows easily by induction. We shall, in fact, prove the irreducibility of the restriction of $S^{k,n}$ to the *special orthogonal group* $SO(n)$ consisting of those orthogonal transformations that have determinant 1 (these transformations are called *rotations* and $SO(n)$ is also known as the *rotation group* on $\mathbf{R}^n$). The group $SO(n-1)$ can be identified is a natural way with a subgroup of $SO(n)$. This can be done by fixing the vector $\mathbf{1} = (0, \ldots, 0, 1)$ in $\mathbf{R}^n$ and considering the subgroup $G \subset SO(n)$ of all rotations leaving $\mathbf{1}$ fixed. Each such rotation effects a change in the first $(n-1)$ coordinates of a point of $\mathbf{R}^n$ and can, therefore, be considered a rotation acting on $\mathbf{R}^{n-1}$. We shall write $SO(n-1) = G \subset SO(n)$.

The theorem will be established if we show that (i) *If* R *is the restriction of the left regular representation of* SO (n) *to a subspace* V *of* $\mathscr{P}^{(k)}$ *then there exists a polynomial* $q \in V$ *that is invariant under the action of* SO (n − 1). *That is,* $q(u^{-1}x) = q(x)$ *for all* $u \in SO(n-1)$; (ii) *If* W *is a subspace of* $\mathscr{H}_n^{(k)}$ *consisting of vectors that are invariant under the action of* SO (n − 1) *then the dimension of* W *is* 1.

If $S^{k,n}$ were not irreducible then $\mathscr{H}_n^{(k)}$ would be the direct sum of (at least) two invariant subspaces. By (i) each of these subspaces must contain a vector invariant under $SO(n-1)$; but this would contradict (ii).

To show (i) we choose an orthormal basis $\{Y_j\}$ of V and, for each pair of points $x, y \in \mathbf{R}^n$, we define

$$Z_x(y) = \sum_j \overline{Y_j(x)}\,Y_j(y)\,. \tag{2.13}$$

Then $(p, Z_x) = \sum_j (p, Y_j)\;Y_j(x) = p(x)$ for all $p \in V$. This means that Z_x is the unique element of V representing the linear functional mapping p into $p(x)$ and, therefore, Z_x is independent of the orthonormal basis we chose. Since S^0 is a unitary transformation on V the functions whose values at $y \in \mathbf{R}^n$ are $Y_j(u^{-1}y)$ also form an orthonormal basis of V. Thus, by (2.13) and the fact that the definition of $Z_x(y)$ is independent of the basis we chose, $Z_{u^{-1}x}(u^{-1}y) = Z_x(y)$ for all u in $SO(n)$. In particular, Z_1 must be invariant under the action of any u in $SO(n-1)$. This proves (i).

To show (ii) we let p be an invariant polynomial under the action of $SO(n-1)$ and we write

$$p(x_1, x_2, \ldots, x_n) = p(x) = \sum_{j=0}^{k} x_n^{k-j} p_j(\xi),$$

where p_j is homogeneous of degree j in the $n-1$ variables $\xi = (x_1, x_2, \ldots, x_{n-1})$. If $u \in SO(n-1)$ and $u^{-1}x = y = (y_1, y_2, \ldots, y_n)$ then $y_n = x_n$. Thus, by our identification of $SO(n-1)$ with a subgroup of $SO(n)$, $(y_1, y_2, \ldots, y_{n-1}) = \eta = u^{-1}\xi$ and

$$\sum_{j=0}^{k} x_n^{k-j} p_j(\xi) = p(x) = p(u^{-1}x) = \sum_{j=0}^{k} x_n^{k-j} p_j(u^{-1}\xi) = \sum_{j=0}^{k} x_n^{k-j} p_j(\eta)$$

for all real numbers x_n. Consequently, $p_j(\xi) = p_j(u^{-1}\xi)$ for all $\xi \in \mathbf{R}^{(n-1)}$ and $u \in SO(n-1)$. But this clearly means that p_j is a *radial function* (i.e. it depends only on $|\xi| = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$) since, if we are given any two points ξ and η with $|\xi| = |\eta|$, there exists a rotation u such that $\eta = u^{-1}\xi$. On the other hand, p_j being homogeneous of degree j, this means that we must have $p_j(\xi) = c_j |\xi|^j = c_j(x_1^2 + \ldots + x_{n-1}^2)^{j/2}$, where c_j is the value of p_j at any point on the surface of the unit sphere $\Sigma_{n-2} = \{\xi \in \mathbf{R}^{(n-1)};\ |\xi| = 1\}$. Since p_j is a polynomial c_j must be zero when j is odd. Thus, after relabeling, we have shown that

$$p(x) = \sum_{0 \leqq j \leqq k/2} c_j x_n^{k-2j}(x_1^2 + \ldots + x_{n-1}^2)^j.$$

On the other hand, since p is harmonic

$$0 = (\Delta p)(x) = \sum_{1 \leqq j \leqq k/2} (\alpha_j c_j + \beta_j c_{j-1}) x_n^{k-2j}(x_1^2 + \ldots + x_{n-1}^2)^{j-1},$$

where $\alpha_j = 2j(n+2j-3)$ and $\beta_j = (k-2j+1)(k-2j+2)$. Since $\alpha_j \neq 0$ for $1 \leqq j \leqq k/2$ this means that

$$c_j = (-1)^j \frac{\beta_1 \beta_2 \ldots \beta_j}{\alpha_1 \alpha_2 \ldots \alpha_j} c_0$$

for $1 \leqq j \leqq k/2$. This shows, therefore, that $p(x)$ is c_0 times the polynomial

$$(2.14) \qquad x_n^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1 \beta_2 \dots \beta_j}{\alpha_1 \alpha_2 \dots \alpha_j} x_n^{k-2j} (x_1^2 + \dots + x_{n-1}^2)^j$$

and (ii) is proved.

COROLLARY (2.15). *The linear space spanned by the class of all polynomials in* $\mathscr{H}_n^{(k)}$, k $= 0, 1, 2, \dots$, *restricted to the surface* $\Sigma_{n-1} = \{ \mathrm{x} \in \mathbf{R}^n : |\mathrm{x}| = 1 \}$ *of the unit sphere in* $\mathbf{R}^n$ *is uniformly dense in the space* $\mathrm{C}(\Sigma_{n-1})$ *of continuous functions on* Σ_{n-1}.

Proof. It follows from the Weierstrass approximation theorem that the linear space spanned by the class of all polymonials in $\mathscr{P}^{(k)}$, $k = 0, 1, \dots$, restricted to Σ_{n-1} is uniformly dense in $C(\Sigma_{n-1})$. But it follows from theorem (2.12) that if $p(x)$ is in $\mathscr{P}^{(k)}$ then

$$p(x) = h(x) + |x|^2 q_1(x) + |x|^4 q_2(x) + \dots + |x|^{2l} q_l(x),$$

where $h \in \mathscr{H}_n^k$ and $q_j \in \mathscr{H}_n^{(k-2j)}$, $1 \leqq j \leqq l \leqq k/2$. If $x \in \Sigma_{n-1}$, therefore, $p(x) = h(x) + q_1(x) + q_2(x) + \dots + q_l(x)$. That is, p is a (finite) sum of elements of $\mathscr{H}_n^{(j)}$, $0 \leqq j \leqq k$. The corollary now follows immediately.

The harmonic homogeneous polynomials of degree k (that is, the members of $\mathscr{H}_n^{(k)}$) are called the *solid spherical harmonics of degree k*. Their restrictions to the surface of the unit sphere Σ_{n-1} are called the *spherical harmonics of degree k* (or, sometimes, the *surface spherical harmonics*). If $p(\xi)$ is such a restriction, because of the homogeneity, we obtain the value of the original function at any point $x = |x| \xi$ in $\mathbf{R}^n$ by multiplying $p(\xi)$ by $|x|^k$. In view of this close relationship between the spaces of solid and surface spherical harmonics of degree k we denote both of them by $\mathscr{H}_n^{(k)}$. It will be clear from the context which of the two spaces $\mathscr{H}_n^{(k)}$ is under discussion. Furthermore, we will systematically use the Greek letters $\xi, \eta, \dots$ to denote points of Σ_{n-1}, while $x, y, \dots$ will continue to denote the general points of $\mathbf{R}^n$. The spherical harmonic Z_ξ, $\xi \in \Sigma_{n-1}$, defined by (2.13) (which was shown to be independent of the choice of orthonormal basis of $\mathscr{H}_n^{(k)}$) is called the *zonal harmonic with pole* ξ. It is clear from our discussion that $c_0 = Z_{\mathbf{1}}(\mathbf{1})$ times the expression (2.14) equals $Z_{\mathbf{1}}(x)$.

The following theorem is a basic tool that will be used in the next section in order to show how the spherical harmonics can be obtained from irreducible representations. Before stating it we observe that $\| Z_1 \| = \sqrt{(Z_1, Z_1)} = \sqrt{Z_1(\mathbf{1})} = a_k$. This follows immediately from the fact that Z_1 represents the linear functional mapping $p \in \mathscr{H}_n^{(k)}$ onto $p(\mathbf{1})$; that is, $(p, Z_1) = p(\mathbf{1})$. For, taking $p = Z_1$, we then must have $\| Z_1 \|^2 = (Z_1, Z_1) = Z_1(\mathbf{1})$.

THEOREM (2.16). *Let* $\{ Y_1, Y_2, \ldots, Y_{d_k} \}$ *be an orthonormal basis of* $\mathscr{H}_n^{(k)}$ *the space of (surface) spherical harmonics of degree* k, *such that* $Y_1 = a_k^{-1} Z_1$. *Then, if* $(t_{ij}(u))$, $u \in SO(n)$, *is the matrix of* $S_u^{k,n}$ *with respect to this basis, we have*

$$Y_j(u\mathbf{1}) = a_k \overline{t_{j1}(u)} = \sqrt{Z_1(\mathbf{1})}\, \overline{t_{j1}(u)} \tag{2.17}$$

for $j = 1, 2, \ldots, d_k$.

Proof. If $p \in \mathscr{H}_n^{(k)}$ is orthogonal to Y_1 we obtain

$$0 = (p, Y_1) = a_k^{-1}(p, Z_1) = a_k^{-1} p(\mathbf{1}).$$

In particular,

(i) $$Y_i(\mathbf{1}) = 0 \quad \textit{for} \quad i = 2, 3, \ldots, d_k.$$

If $v \in SO(n)$ then the matrix $(t_{ij}(v))$ of $S_v^{k,n}$ is given by

$$(S_v^{k,n} Y_j)(\xi) = Y_j(v^{-1}\xi) = \sum_{i=1}^{d_k} t_{ij}(v)\, Y_i(\xi), \quad 1 \leqq j \leqq d_k.$$

Thus, putting $\xi = \mathbf{1}$ and using (i), we obtain

$$Y_j(v^{-1}\mathbf{1}) = t_{1j}(v)\, Y_1(\mathbf{1}) = a_k \overline{t_{j1}(v^{-1})}, \quad 1 \leqq j \leqq d_k.$$

Letting $u = v^{-1}$ this equality reduces to relation (2.17) and the theorem is proved.

It is not hard to evaluate the constant $a_k^2 = Z_1(\mathbf{1})$. In fact, let

$$q(x) = x_n^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1 \beta_2 \ldots \beta_j}{\alpha_1 \alpha_2 \ldots \alpha_j} x_n^{k-2j} (x_1^2 + x_2^2 + \ldots + x_{n-1}^2)^j$$

be the polynomial (2.14). We showed that $Z_1(\mathbf{1})\, q(x) = Z_1(x)$. Thus,

$$Z_1(\mathbf{1}) = (Z_1, Z_1) = (a_k^2 q, a_k^2 q) = [Z_1(\mathbf{1})]^2 (q, q).$$

This shows that $a_k^2 = Z_1(\mathbf{1}) = 1 / (q, q)$. On the other hand, the inner product (q, q) is easily evaluated once we observe, after an easy calculation, that the polynomials $x_n^{k-2j}(x_1^2 + x_2^2 + \ldots + x_{n-1}^2)^j$ are mutually orthogonal and the square of their norm is $\alpha_1 \alpha_2 \ldots \alpha_j / \beta_1 \beta_2 \ldots \beta_j$. Hence,

$$a_k^{-2} = 1/Z_1(\mathbf{1}) = 1 + \sum_{1 \leqq j \leqq k/2} \frac{\beta_1 \beta_2 \ldots \beta_j}{\alpha_1 \alpha_2 \ldots \alpha_j}, \tag{2.18}$$

where $\alpha_j = 2j(n + 2j - 3)$ and $\beta_j = (k - 2j + 1)(k - 2j + 2)$. It can be shown that the last expression equals

$$\prod_{j=0}^{k-1} \left(\frac{2j + n - 2}{j + n - 2} \right);$$

thus, we also have

$$a_k^{-2} = \prod_{j=0}^{k-1} \frac{2j + n - 2}{j + n - 2}. \text{[1]} \tag{2.19}$$

In the next section we shall characterize those irreducible representations of $SO(n)$ that are equivalent to $S^{k,n}$. These will be the representations of class 1 (to be defined in § 3). We shall show that the spaces spanned by the first column of the matrix of these representations with respect to certain orthonormal bases are the same whenever two representations are equivalent. Consequently, we can define $Y_j, j = 1, 2, \ldots, d_k$, by formula (2.17) when $(t_{ij}(u))$ is such a matrix and obtain the spaces $\mathscr{H}_n^{(k)}$ directly from the general theory of representations of $SO(n)$.

§ 3. Representations of Class 1 and Spherical Harmonics

In the course of the proof of theorem (2.12) we showed that there was precisely a one dimensional subspace of $\mathscr{H}_n^{(k)}$ whose points invariant under the action of $S^{k,n}$ restricted to $SO(n-1)$. As we shall see, it is this property

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1) When $n = 4$, for example, $a_k^2 = (k+1)2^{-k}$. For $n = 6$, $a_k^2 = 6(k+2)(k+3)2^{-k}$. The fact that $a_k^2 = Z_1(\mathbf{1}) \leqq 1$ can be shown without any calculation. Equation (2.13) defined a "zonal harmonic" for any subspace of $\mathscr{P}^{(k)}$. If, in this definition, we use the orthonormal basis $\left\{\sqrt{\binom{k}{\alpha}}\, x^{\alpha}\right\}$, $\| \alpha \| = k$, we obtain $(x \cdot y)^k$. The value of this function at $x = \mathbf{1} = y$ is obviously 1. If, on the other hand, we use an orthonormal basis that is a continuation of an orthonormal basis of $\mathscr{H}^{(k)}$ we clearly have $Z_1(\mathbf{1}) \leqq (\mathbf{1} \cdot \mathbf{1})^k = 1$.