§3. Representations of Class 1 and Spherical Harmonics

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This shows that $a_k^2 = Z_1(\mathbf{1}) = 1 / (q, q)$. On the other hand, the inner product (q, q) is easily evaluated once we observe, after an easy calculation, that the polynomials $x_n^{k-2j}(x_1^2 + x_2^2 + \ldots + x_{n-1}^2)^j$ are mutually orthogonal and the square of their norm is $\alpha_1 \alpha_2 \ldots \alpha_j / \beta_1 \beta_2 \ldots \beta_j$. Hence,

$$(2.18) \qquad a_k^{-2} = 1/Z_1(\mathbf{1}) = 1 + \sum_{1 \leqq j \leqq k/2} \frac{\beta_1 \beta_2 \ldots \beta_j}{\alpha_1 \alpha_2 \ldots \alpha_j},$$

where $\alpha_j = 2j(n+2j-3)$ and $\beta_j = (k-2j+1)(k-2j+2)$. It can be shown that the last expression equals

$$\prod_{j=0}^{k-1} \binom{2j+n-2}{j+n-2};$$

thus, we also have

$$(2.19) \qquad a_k^{-2} = \prod_{j=0}^{k-1} \frac{2j+n-2}{j+n-2}.$$ [1]

In the next section we shall characterize those irreducible representations of $SO(n)$ that are equivalent to $S^{k,n}$. These will be the representations of class 1 (to be defined in § 3). We shall show that the spaces spanned by the first column of the matrix of these representations with respect to certain orthonormal bases are the same whenever two representations are equivalent. Consequently, we can define $Y_j, j = 1, 2, \ldots, d_k$, by formula (2.17) when $(t_{ij}(u))$ is such a matrix and obtain the spaces $\mathscr{H}_n^{(k)}$ directly from the general theory of representations of $SO(n)$.

§ 3. Representations of Class 1 and Spherical Harmonics

In the course of the proof of theorem (2.12) we showed that there was precisely a one dimensional subspace of $\mathscr{H}_n^{(k)}$ whose points invariant under the action of $S^{k,n}$ restricted to $SO(n-1)$. As we shall see, it is this property

1) When $n = 4$, for example, $a_k^2 = (k+1)2^{-k}$. For $n = 6$, $a_k^2 = 6(k+2)(k+3)2^{-k}$. The fact that $a_k^2 = Z_1(\mathbf{1}) \leqq 1$ can be shown without any calculation. Equation (2.13) defined a "zonal harmonic" for any subspace of $\mathscr{P}^{(k)}$. If, in this definition, we use the orthonormal basis $\left\{\sqrt{\binom{k}{\alpha}}\, x^\alpha\right\}$, $\|\alpha\| = k$, we obtain $(x \cdot y)^k$. The value of this function at $x = \mathbf{1} = y$ is obviously 1. If, on the other hand, we use an orthonormal basis that is a continuation of an orthonormal basis of $\mathscr{H}^{(k)}$ we clearly have $Z_1(\mathbf{1}) \leqq (\mathbf{1} \cdot \mathbf{1})^k = 1$.

that will enable us to identify those irreducible representations of $SO(n)$ that are equivalent to $S^{k,n}$. In order to do this we shall study, more generally, those representations T of a compact group G having the property that there exists a compact subgroup K and a subspace W of the Hilbert space on which T acts such that $T(u)$, for $u \in K$, is the identity transformation when restricted to W. Before doing this, however, we would like to show that there is a close connection between $SO(n)$ and Σ_{n-1}.

To begin with, it is not hard to show that the space $SO(n) / SO(n-1)$ of left cosets $[u] = \{ uw : u \in SO(n), w \in SO(n-1) \}$ can be identified in a natural way with the surface of the unit sphere Σ_{n-1}. The topology on this space is the one induced by the projection $u \to [u]$ of $SO(n)$ into $SO(n) / SO(n-1)$ (i.e. a set in this last space is open if and only if its inverse image is open). Given $u \in SO(n)$ let $x_u = u\mathbf{1}$; if $v \in [u]$ then $v\mathbf{1} = u\mathbf{1}$ since $u^{-1} v \in SO(n-1)$ and, thus, $v\mathbf{1} = (uu^{-1}) v\mathbf{1} = u(u^{-1} v\mathbf{1}) = u\mathbf{1}$. Consequently, the mapping $\Phi : [u] \to x_u = u\mathbf{1}$ is well defined. Moreover, it is clear that Φ is continuous, one to one and onto Σ_{n-1}. Since $SO(n) / SO(n-1)$ is compact it follows that Φ is a homeomorphism.

Secondly, the Haar measure of $SO(n)$ can be used in order to obtain the ordinary Lebesgue measure on Σ_{n-1}. This follows from the following result.

THEOREM (3.1). *If* f *is a continuous function on* Σ_{n-1} *and* $\xi_0 \in \Sigma_{n-1}$, *then*

$$\int_{\Sigma_{n-1}} f(\xi)\, d\xi = \int_{SO(n)} f(u\xi_0)\, du ,$$

where $d\xi$ *is the element of normalized Lebesgue measure on* Σ_{n-1} *(that is,* $\int_{\Sigma_{n-1}} d\xi = 1$*) and* du *is the element of normalized Haar measure on the group* SO (n).

Proof. The only property of Lebesgue measure on Σ_{n-1} that we need is that it is invariant under the action of rotations. Thus, first using this property, then the fact that Haar measure is normalized and Fubini's theorem we have

$$\int_{\Sigma_{n-1}} f(\xi)\, d\xi = \int_{\Sigma_{n-1}} f(u\xi)\, d\xi = \int_{SO(n)} \Big\{ \int_{\Sigma_{n-1}} f(u\xi)\, d\xi \Big\}\, du$$

$$= \int_{\Sigma_{n-1}} \Big\{ \int_{SO(n)} f(u\xi)\, du \Big\}\, d\xi .$$

Let u_0 be a rotation such that $u_0 \xi_0 = \xi$; then, since the compact group $SO(n)$ must be unimodular, the last integral equals

$$\int_{\Sigma_{n-1}} \{ \int_{SO(n)} f(uu_0 \xi_0)\, du \}\, d\xi = \int_{\Sigma_{n-1}} \{ \int_{SO(n)} f(u\xi_0)\, du \}\, d\xi$$
$$= \{ \int_{SO(n)} f(u\xi_0)\, du \} \{ \int_{\Sigma_{n-1}} d\xi \} = \int_{SO(n)} f(u\xi_0)\, du .\text{[1]}$$

We now turn to the general case described at the beginning of § 3. That is, we suppose G is a compact group and K a closed subgroup. Suppose T is a representation of G acting on a Hilbert space H and $W \subset H$ the subspace of all those vectors in H that are invariant under the action of K; that is,

$$W = \{ s \in H;\ T_u s = s \text{ for } u \in K \}.$$

For example, as was mentioned briefly at the beginning of this section, when $G = SO(n)$, $K = SO(n-1)$, $T = S^{k,n}$ and $H = \mathscr{H}_n^{(k)}$ we showed in the course of the proof of theorem (2.12) that W is the one dimensional subspace generated by the zonal harmonic Z_1.

The restriction of T to K is a representation of this subgroup that acts on H. If we choose an orthonormal basis of H that is an extension of an orthonormal basis of W, then the matrix of $T(u)$ with respect to this basis has the form

$$\begin{pmatrix} I_W & 0 \\ 0 & \tilde{T}(u) \end{pmatrix}$$

for all $u \in K$, where I_W is the matrix of the identity operator on W. The mapping $\tilde{T}: u \to \tilde{T}(u) = \tilde{T}_u$ is a (matrix valued) representatoin of K acting on $W^{\perp}$. Let ν be the normalized Haar measure of the group K. Then, if dimension d of H is greater than 1 it follows from the orthogonality relations of theorem (1.3) that

$$\text{(3.2)} \qquad \int_K T(u)\, d\nu(u) = \begin{pmatrix} I_W & 0 \\ 0 & 0 \end{pmatrix};$$

or, equivalently, if $(t_{ij}(u))$ is the matrix of $T(u)$ with respect to the above mantioned basis,

$$\text{(3.2')} \qquad \int_K t_{ij}(u)\, d\nu(u) = \begin{cases} \delta_{ij} & \text{if } \ i,j \leqq c = \dim W . \\ 0 \text{ otherwise} \end{cases}$$

[1]) The reader should observe that this proof obviously extends to the case when $SO(n)$ is replaced by anyone of its compact subgroups that act transitively on Σ_{n-1}.

If $v \in G$ it follows from (3.2) and the fact that T is a representation that

(3.3)

$$\int_K T(vu)\,dv(u) = T(v)\int_K T(u)\,dv(u) = \begin{pmatrix} t_{11}(v) \dots t_{1c}(v)\; 0 \dots 0 \\ t_{21}(v) \dots t_{2c}(v)\; 0 \dots 0 \\ \dots\dots\dots\dots\dots \\ t_{d1}(v) \dots t_{dc}(v)\; 0 \dots 0 \end{pmatrix}.$$

Suppose f is a continuous functions on G that is constant on the left cosets vK; that is, $f(vu) = f(v)$ for all $u \in K$. Then,

$$f(v) = \int_K f(v)\,dv(u) = \int_K f(vu)\,dv(u)\,.$$

On the other hand, an application of Minkowski's integral inequality and (1.5) gives us

$$\int_K f(vu)\,dv(u) = \sum_{\alpha \varepsilon \mathscr{A}} \Big(\sum_{i,j=1}^{d_\alpha} c^\alpha_{ij} \int_K t^\alpha_{ij}(vu)\,dv(u)\Big),$$

where $\{T^\alpha\} = \{(t^\alpha_{ij})\}$ is a complete system of irreducible matrix valued representations of G and the convergence is in the norm of $L^2(G)$. Thus, from (3.3) we have

(3.4) $$f(v) = \sum_{\alpha \varepsilon \mathscr{A}} \Big(\sum_{i=1}^{d_\alpha} \sum_{j=1}^{c_\alpha} c^\alpha_{ij}\, t^\alpha_{ij}(v)\Big)$$

where c_α is the dimension of $W_\alpha = \{s \in H^\alpha : T^\alpha_u s = s \text{ for } u \in K\}$ and the convergence is, again, in the norm of $L^2(G)$.

When $K = G$ then the spaces W_α must be zero dimensional. On the other hand, $c_\alpha = d_\alpha$ if K consists of only the identity element of G. When $c_\alpha = 1$ the representation T^α is said to be of *class* 1 *with respect to* K. The following theorem shows that, if $G = SO(n)$ and $K = SO(n-1)$, then an irreducible representation of G is either of class 1 with respect to K or there are no vectors in the space on which the representation acts that are invariant under the action of K.

THEOREM (3.5). *Suppose* T *is an irreducible representation of* SO (n), n $\geqq$ 3, *acting on the Hilbert space* H *and* W $= \{$s $\in$ H: T$_u$ s $=$ s *for* u $\in$ SO (n$-$1)$\}$. *Then the dimension of* W *is eigher* 0 *or* 1.

Proof. Given $v \in SO(n)$ we claim that there exists $w \in SO(n-1)$ such that $wv\mathbf{1} = v^{-1}\mathbf{1}$. If $v \in SO(n-1)$ we can take w to be the identity.

If $v \notin SO(n-1)$ we first observe that $v\mathbf{1} - v^{-1}\mathbf{1}$ is orthogonal to $\mathbf{1}$ because

$$v\mathbf{1}\cdot\mathbf{1} = \mathbf{1}\cdot v^{*}\mathbf{1} = \mathbf{1}\cdot v^{-1}\mathbf{1} = v^{-1}\mathbf{1}\cdot\mathbf{1}.$$

Since $n \geqq 3$ we can construct a two dimensional subspace V of $\mathbf{R}^n$ spanned by $v\mathbf{1} - v^{-1}\mathbf{1}$ and another vector orthogonal to $\mathbf{1}$. If we rotate this space about $\mathbf{1}$ by π radians and leave the orthogonal complement of the span of $\mathbf{1}$ and V pointwise fixed, we obtain a transformation $w \in SO(n-1)$ with the desired property.

If we put $u_1 = vwv$ and $u_2 = w^{-1}$ then $u_1, u_2 \in SO(n-1)$ and $v = u_1 v^{-1} u_2$. Thus,

(i) $$T_v = T_{u_1} T_{v^{-1}} T_{u_2}.$$

Suppose $\dim W \geqq 2$. Then we can find two vectors, e_1 and e_2, such that $(e_i, e_j) = \delta_{ij}$ and $T_u e_j = e_j$ for $i, j = 1, 2$ and $u \in SO(n-1)$. Let $t_{ij}(u)$ be the entries of the matrix of T_u with respect to an orthonormal basis of H having e_1 and e_2 as its first and second elements. Then $t_{ij}(u) = \delta_{ij}$ if either i or j is 1 or 2 and $u \in SO(n-1)$. This fact, together with (1.2) imply that $t_{ij}(vu) = t_{ij}(v)$ and $t_{ij}(uv) = t_{ij}(v)$ when $i, j = 1, 2$, $u \in SO(n-1)$ $v \in SO(n)$. From equality (i), therefore, we have $t_{ij}(v) = \overline{t_{ji}(v)}$ for $i, j = 1, 2$ and all $v \in SO(n)$.

In particular,

(ii) $$t_{21}(v) = \overline{t_{12}(v)}$$

and

(iii) $$t_{11}(v) = \overline{t_{11}(v)}$$

for all $v \in SO(n)$.

If H_1 is the space generated by $t_{11}, t_{21}, \ldots, t_{d1}$ it then follows from theorem (1.6) that the span of the left translates of t_{11} is again H_1 (otherwise this span would be a proper invariant subspace of H_1). Thus, there exist a finite number of rotations $u_1, u_2, \ldots, u_m$ and constant $c_1\, c_2, \ldots, c_m$ such that

$$t_{21}(u) = \sum_{j=1}^{m} c_j\, t_{11}(u_j^{-1} u)$$

for all $u \in SO(n)$. But, by theorem (1.3)

$$\int_{SO(n)} t_{21}(u)\, t_{12}(u)\, du = \int_{SO(n)} \overline{t_{12}(u)}\, t_{12}(u)\, du = 1/d.$$

On the other hand, again using the orthogonality relations of theorem (1.3),

$$\int_{SO(n)} t_{11}(u_j^{-1} u)\, t_{12}(u)\, du = \int_{SO(n)} \overline{t_{11}(u_j^{-1} u)\, t_{12}(u)}\, du =$$

$$= \int_{SO(n)} \sum_{l=1}^{d} t_{1l}(u_j^{-1})\, t_{l1}(u)\, t_{12}(u)\, du =$$

$$= \sum_{l=1}^{d} \overline{t_{1l}(u_j^{-1})} \int_{SO(n)} \overline{t_{l1}(u)\, t_{12}(u)}\, du = 0\,.$$

Hence,

$$\int_{SO(n)} t_{21}(u)\, t_{12}(u)\, du = \sum_{j=1}^{m} c_j \int_{SO(n)} t_{11}(u_j^{-1} u)\, t_{12}(u)\, du = 0\,.$$

We therefore obtain the contradiction $1 / d = 0$ and the theorem is proved.

If $\{ T^{\alpha} \}$, $\alpha \in \mathscr{A}$, is a complete system of irreducible matrix valued representations of $SO(n)$, $n \geqq 3$, some of the T^{α}'s will be of class 1 with respect to $SO(n-1)$. The rest of the T^{α}'s will act on Hilbert spaces having no non-zero vectors invariant under the action of $SO(n-1)$. Let $\mathscr{A}_1 \subset \mathscr{A}$ be the set of α such that T^{α} is of class 1 with respect to $SO(n-1)$.

We fix such a complete system $\{ T^{\alpha} \}$ of irreducible matrix valued representations of $SO(n)$. Suppose T is a representation equivalent to T^{α} for some $\alpha \in \mathscr{A}_1$; that is, T is of class 1 with respect to $SO(\mathrm{n}-1)$. If H is the Hilbert space on which T acts then there exists a 1 dimensional sub-space that is invariant under the action of $SO(n-1)$. Let $\{ Y_1, Y_2, \ldots, Y_d \}$ be an orthonormal basis of H such Y_1 spans this 1 dimensional space and $(t_{ij}(u))$ the matrix of T_u with respect to this basis. Then, by (3.3)

$$\int_{SO(n-1)} T(vu)\, d\nu(u) = \begin{pmatrix} t_{11}(v)\ 0 \ldots 0 \\ t_{21}(v)\ 0 \ldots 0 \\ \ldots\ldots\ldots\ldots\ldots \\ t_{d1}(v)\ 0 \ldots 0 \end{pmatrix},$$

whenever $v \in SO(n)$

In particular, if χ is the character of T (see § 1), we have

$$\int_{SO(n-1)} \chi(vu)\, du = t_{11}(v) \tag{3.6}$$

for all $v \in SO(n)$. Since the character χ is the same for all members of the class $[T]$ of representations equivalent to T, (3.6) gives us a definition of t_{11} that does not depend on the representative we choose from $[T]$. The same must therefore be true of the vector space spanned by the left translates of t_{11}. By theorem (1.6) this vector space must be the linear span H_1 of the elements $t_{11}, t_{21}, \ldots, t_{d1}$ of the first column of the matrix (t_{ij}) (for, as was argued in the proof of theorem (3.5), if this were not the case H_1 would have a proper invariant subspace). In the proof of (1.6) we showed, moreover, that the restriction to H_1 of the left regular representation of $SO(n)$ is equivalent to the representation $\bar{T}$ whose matrix with respect to $\{Y_1, Y_2, \ldots, Y_d\}$ is $\overline{(t_{ij})}$. In fact, we showed that this restriction $R^{(1)}$ equals $L\bar{T}L^{-1}$, where L is the isometric linear transformation of H onto H_1 mapping Y_i onto $\sqrt{d}\,t_{i1}$, $1 \leqq i \leqq d$ (see footnote at the end of § 1).

We can apply these arguments to the representation $S^{k,n}$ acting on $\mathscr{H}_n^{(k)}$ since it is of class 1 with respect to $SO(n-1)$; in fact, for Y_1 we can choose $a_k^{-1} Z_1$ where $a_k = \sqrt{Z_1(\mathbf{1})} = \sqrt{(Z_1, Z_1)}$ (see theorem (2.16)). In particular there exists $\alpha = \alpha_k \in \mathscr{A}_1$ such that $S^{k,n}$ is equivalent to T^α. We then obtain the same function t_{11}^α from (3.6) by taking $\chi = \chi^\alpha$ to be either the character of $S^{k,n}$ or of T^α. Thus, $S^{k,n}$ and T^α are (isometrically) equivalent to the restriction of the left regular representation of $SO(n)$ to the vector space generated by the left translates of $\overline{t_{11}^\alpha} = \bar{t}^\alpha$. It follows from (2.17) and (3.6) that

$$Z_1(v\mathbf{1}) / Z_1(\mathbf{1}) = \overline{t^\alpha(v)} = \int_{SO(n-1)} \overline{\chi^\alpha(vu)}\, du$$

for all $v \in SO(n)$. In equation (2.13), which was used in order to define Z_1, we could have chosen an orthonormal basis $\{Y_1, Y_2, \ldots, Y_{d_k}\}$ of $\mathscr{H}_n^{(k)}$ that consists of real valued functions (recall that the sum on the right is independent of the choice of orthonormal basis); we see, therefore, that Z_1 must be real valued. Consequently, we can omit the bars denoting complex conjugation in the last equality and we obtain

$$Z_1(v\mathbf{1}) / Z_1(\mathbf{1}) = t^\alpha(v) = \int_{SO(n-1)} \chi^\alpha(vu)\, du \tag{3.7}$$

for all $v \in SO(n)$.

It follows that a function p on Σ_{n-1} is a spherical harmonic belonging to $\mathscr{H}_n^{(k)}$ if and only if

$$p(u\mathbf{1}) = F(u), \tag{3.8}$$

where F is a finite linear combination of left translates of t^α.

Suppose $p, q \in \mathscr{H}_n^{(k)}$. In § 2 we defined their inner product (p, q)

(see (2.6), (2.6′) and (2.6″)). Since $\mathscr{H}_n^{(k)} \subset L^2(\Sigma_{n-1})$ we can also form the inner product that $L^2(\Sigma_{n-1})$ induces on $\mathscr{H}_n^{(k)}$:

$$< p, q > \; = \int_{\Sigma_{n-1}} p(\xi)\, \overline{q(\xi)}\, d\xi\,. \tag{3.9}$$

It is not hard to show that these two inner products differ only by a multiplicative constant:

$$< p, q > \; = \; A_k(p, q)\,, \tag{3.10}$$

where $A_k = Z_1(\mathbf{1}) / d_k = a_k^2 / d_k$.

In order to show this we choose the orthonormal basis $\{ Y_1, Y_2, \ldots, Y_{d_k} \}$ of theorem (2.16), let $p = \Sigma\, b_j Y_j$ and $q = \Sigma\, c_j Y_j$. Then,

$$(p, q) = \sum_{j=1}^{d_k} b_j \bar{c}_j\,.$$

But by (2.16), (3.1) and (1.3),

$$< p, q > = \int_{\Sigma_{n-1}} p(\xi)\, \overline{q(\xi)}\, d\xi = \int_{SO(n)} a_k^2 \Big(\sum_{j=1}^{d_k} b_j\, t_{j1}(u) \Big) \Big(\overline{\sum_{j=1}^{d_k} c_j t_{j1}(u)} \Big)\, du =$$

$$= a_k^2\, d_k^{-1} \sum_{j=1}^{d_k} b_j \bar{c}_j = a_k^2\, d_k^{-1}\,(p, q)\,.$$

In the discussion following (2.13) we showed that $Z_{v\xi}(v\eta) = Z_\xi(\eta)$ for all $v \in SO(n)$ and $\xi, \eta \in \Sigma_{n-1}$. Thus,

$$< Z_{v\xi}, Z_{v\xi} > \; = \int_{\Sigma_{n-1}} | Z_{v\xi}(\eta) |^2\, d\eta = \int_{\Sigma_{n-1}} | Z_{v\xi}(v\eta) |^2\, d\eta =$$

$$= \int_{\Sigma_{n-1}} | Z_\xi(\eta) |^2\, d\eta = \; < Z_\xi, Z_\xi > .$$

Consequently,

$$< Z_\xi, Z_\xi > \; = \; < Z_\eta, Z_\eta > \tag{3.11}$$

for all $\xi, \eta \in \Sigma_{n-1}$. Using the fact that $p(\eta) = (p, Z_\eta)$ for $p \in \mathscr{H}_n^{(k)}$, (3.10), Schwarz's inequality and (3.11) we obtain

$$| Z_\xi(\eta) | = | (Z_\xi, Z_\eta) | = A_k^{-1} \, | < Z_\xi, Z_\eta > | \leqq$$

$$\leqq A_k^{-1} \sqrt{< Z_\xi, Z_\xi > < Z_\eta, Z_\eta >} =$$

$$= A_k^{-1} < Z_\xi, Z_\xi > \; = A_k^{-1} A_k (Z_\xi, Z_\xi) = (Z_1, Z_1) = Z_1(\mathbf{1})\,.$$

We have shown that

$$|Z_\xi(\eta)| \leqq Z_1(\mathbf{1}) \tag{3.12}$$

for all $\xi, \eta \in \Sigma_{n-1}$.

It is not hard to show that each representation T^α, $\alpha \in \mathscr{A}_1$, is equivalent to one of the representations $S^{k,n}$, for some $k = 0, 1, \ldots$. We assume that $t^\alpha_{11}(v) = t^\alpha(v)$ is the function defined by (3.7); equivalently, we can assume that $(t^\alpha_{ij}(v))$ is the matrix of the representation $T^\alpha(v)$ with respect to an orthonormal basis of H^α whose first element is invariant under $SO(n-1)$. We claim that, under these conditions, the system

$$\bigcup_{\alpha \in \mathscr{A}_1} \{\sqrt{d_\alpha}\, t^\alpha_{11}, \ldots \sqrt{d_\alpha}\, t^\alpha_{d_\alpha 1}\} \tag{3.13}$$

is a complete orthonormal system for the class of functions f in $L^2(SO(n))$ that are constant on the left cosets $v\, SO(n-1)$; that is, $f(vu) = f(v)$ for all $u \in SO(n-1)$. This follows from (3.4). In fact we have

$$f(v) = \sum_{\alpha \in \mathscr{A}_1} \sum_{i=1}^{d_\alpha} c^\alpha_{i1}\, t^\alpha_{i1}(v) \tag{3.14}$$

for all such functions f (the convergence is in L^2 and the coefficients c^α_{i1} are those introduced in (1.5)). If we consider those indices $\alpha \in \mathscr{A}_1$ that correspond to some $k = 0, 1, 2, \ldots$ in the manner described above (i.e. $S^{k,n}$, being of class 1, must be equivalent to one of the representations $T^\alpha = T^{\alpha k}$ with $\alpha \in \mathscr{A}_1$) we obtain a subcollection of the orthonormal system (3.13). On the other hand, it follows immediately from (2.15) and (3.1) that this subcollection must consist of the entire complete system (3.13).

We collect these various results in the following theorem:

THEOREM (3.15). *Suppose* $\{T^\alpha\}$, $\alpha \in \mathscr{A}$, *is a complete system of irreducible representations of* SO (n) *and* $\mathscr{A}_1 \subset \mathscr{A}$ *is the set of all* $\alpha \in \mathscr{A}$ *such that* T^α *is of class* 1 *with respect to* SO (n−1). *We can then find a one to one correspondence* $k \leftrightarrow \alpha_k$ *between the non-negative integers* $k = 0, 1, 2, \ldots$ *and the members* α $(=\alpha_k)$ *of* $\mathscr{A}_1$ *such that* $S^{k,n}$ *and* $T^{\alpha k}$ *are equivalent. If* χ_k *is the character of* $S^{k,n}$ *(or* $T^{\alpha k}$*) then the zonal harmonic* Z_1 *of* $\mathscr{H}_n^{(k)}$ *is real valued and satisfies*

$$a_k^{-2} Z_1(v\mathbf{1}) = \int_{SO(n-1)} \chi_k(vu)\, du = t^{(k)}(v) \tag{3.16}$$

for all $v \in SO(n)$. *A function* p *on* Σ_{n-1} *is a spherical harmonic of degree* k *if and only if* $p(u\mathbf{1}) = F(u)$, $u \in SO(n)$, *where* F *is a finite linear combina-*

tion of left translates of $t^{(k)}$. *If* (p, q) *is the inner product of* p *and* q *in* $\mathscr{H}_n^{(k)}$ *introduced in* § 2 *and*

$$< p, q > = \int_{\Sigma_{n-1}} p(\xi)\, \overline{q(\xi)}\, d\xi$$

is the inner product of p *and* q *regarded as members of* $L^2(\Sigma_{n-1})$ *then*

$$< p, q > = A_k(p, q) \tag{3.17}$$

where $A_k = Z_1(\mathbf{1}) / d_k = a_k^2 / d_k$. *If* $p \in \mathscr{H}_n^{(k)}$ *and* $q \in \mathscr{H}_n^{(j)}$ *with* $k \neq j$ *then* $< p, q > = 0$. *Suppose* $\{ Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_{d_k}^{(k)} \}$ *is an orthonormal basis of* $\mathscr{H}_n^{(k)}$ *then*

$$\bigcup_{k=0}^{\infty} \{ Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_{d_k}^{(k)} \}$$

is a complete orthonormal system in $L^2(\Sigma_{n-1})$. *The zonal harmonic* Z_ξ, $\xi \in \Sigma_{n-1}$, *is less than or equal to* $Z_1(\mathbf{1})$ *in absolute value.*

Perhaps the only fact we did not explicitly prove is that $< p, q > = 0$ when $p \in \mathscr{H}_n^{(k)}$ anc $q \in \mathscr{H}_n^{(j)}$ with $k \neq j$. But this is an easy consequence of the Peter-Weyl theorem (1.3) and theorem (3.1) since $\int_{SO(n)} F(u)\, \overline{G(u)}\, du = 0$ when F is a finite linear combination of left translates of $t^{(k)}$ and G a finite linear combination of left translates of $t^{(j)}$ (by (1.2) such left translates are linear combinations of entries in the first column of a matrix of the representation with respect to some orthonormal system whose first element is invariant under $SO(n-1)$). We have not considered the problem of determining the degree of homogeneity k from a given irreducible representation T^α of class 1 with respect to $SO(n-1)$. Perhaps the easiest way of doing this is by observing that the dimension of the space H^α on which T^α acts must be the same as that of $\mathscr{H}_n^{(k)}$ when T^α and $S^{k,n}$ are equivalent. But the dimension d_k of $\mathscr{H}_n^{(k)}$ can be easily calculated in terms of k. From theorem (2.12) and the discussion preceeding it, we see that $\mathscr{P}^{(k)}$ is the direct sum of $\mathscr{H}_n^{(k)}$ and $|x|^2 \mathscr{P}^{(k-2)}$. Since this last space obviously has the same dimension as $\mathscr{P}^{(k-2)}$ it follows that $d_k = \dim \mathscr{P}^{(k)} - \dim \mathscr{P}^{(k-2)}$. By an easy combinatorial argument (see Stein and Weiss [10], Chapter IV, §) we can show that

$$\dim \mathscr{P}^{(k)} = \binom{n+k-1}{k} \tag{3.18}$$

Thus, for $n \geqq 3$,

$$d_k = \dim \mathscr{H}_n^{(k)} = \frac{(k+n-3)!\,(2k+n-2)}{(n-2)!\qquad k!}. \tag{3.19}$$

Theorem (3.15) shows us how the spherical harmonics we introduced in § 2 can be obtained from the general theory of representations of compact groups applied to $SO(n)$. We have also obtained several properties of these spherical harmonics by using simple arguments based on this general theory. We claim that essentially all the well-known classical facts concerning these special functions can be obtained by equally simple arguments. In the next section we justify this claim by deriving a number of important results in the theory of spherical harmonics. Our arguments will again be based on the general theory of representations of compact groups.

§ 4. Some properties of spherical harmonics

The zonal harmonics Z_1 are often expressed in terms of certain polynomial functions $P_n^{(k)}$ restricted to the interval $[-1,1]$ that are called the *ultra spherical* (or Gegenbauer) polynomials. We have already obtained such an expression in § 2. In fact let

$$(4.1) \qquad P^{(k)}(t) = a_k^2\Big(t^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1 \beta_2 \dots \beta_j}{\alpha_1 \alpha_2 \dots \alpha_j} t^{k-2j}(1-t^2)^j\Big)$$

for $-1 \leqq t \leqq 1, \alpha_j = 2j(2j+n-3)$, $\beta_j = (k-2j+1)(k-2j+2)$ and $a_k^2 = Z_1(\mathbf{1})$. If $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \Sigma_{n-1}$ and we put $t = \xi_n$, so that $1 - t^2 = \xi_1^2 + \dots + \xi_{n-1}^2$, the expression in parenthesis becomes the polynomial (2.14) exaluated at ξ. The observation we made in the paragraph following the proof of Corollary (2.15) is equivalent to the fact $Z_1(\xi)$ and $P^{(k)}(t)$ are equal. Writing $t = \xi \,.\, \mathbf{1}$ this equality becomes

$$(4.2) \qquad Z_1(\xi) = P^{(k)}(\xi \cdot \mathbf{1}) .$$

Usually, the ultraspherical polynomials are introduced in one of two ways. One method is to apply the Gram-Scmidt process to the powers $1, t, t^2, \dots$ restricted to the interval $[-1, 1]$ with respect to the inner product

$$(4.3) \qquad (f, g) = \int_{-1}^{1} f(t)\overline{g(t)}(1-t^2)^{\frac{n-3}{2}} dt .$$

Another definition of the polynomials $P^{(k)}$ involves the k^{th} derivative of $(1-t^2)^{(2k+n-3)/2}$:

$$(4.4) \qquad P^{(k)}(t) = \alpha_{k,n}(1-t^2)^{(3-n)/2} \frac{d^k}{dt^k}(1-t^2)^{(n+2k-3)/2}$$