§5. Special results for n=4

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§ 5. Special results for $n = 4$

In the literature (see in particular Bateman [1] Vol. 2 § 11.6) especially elegant formulas are given in the four dimensional case. These formulas can be obtained by using $SU(2)$ as a group acting transitively on Σ_3.

We begin by identifying $\mathbf{R}^4, \mathbf{C}^2, \mathbf{R}^+ \times SU(2) = \{ ru \colon r = a \text{ positive real, and } u \in SU(2) \}$, via the following maps.

$$x = (x_1, x_2, x_3, x_4) \leftrightarrow (x_1 + ix_2, x_3 + ix_4) = (\chi_1, \chi_2)$$

$$(\chi_1, \chi_2) \leftrightarrow -i \begin{pmatrix} -\bar{\chi}_2, \chi_1 \\ \bar{\chi}_1, \chi_2 \end{pmatrix} = i \,|\, x \,|\, \begin{pmatrix} -\bar{\chi}_2', \chi_1' \\ \bar{\chi}_1', \chi_2' \end{pmatrix} = |\, x \,|\, u_x$$

It is easily checked that

$$u_x = -i \begin{pmatrix} -\chi_2', \chi_1' \\ \bar{\chi}_1', \bar{\chi}_2' \end{pmatrix}$$

belongs to $SU(2)$.

Clearly, when $|x| = 1$ the correspondence $x \leftrightarrow u_x$ permits us to identify Σ_3 with $SU(2)$. We chose this map $x \to u_x$ in order to obtain (identifying $\mathbf{1}$ with (o, i)).

$$u_x \mathbf{1} = -i \begin{pmatrix} -\bar{\chi}_2, \chi_1 \\ \bar{\chi}_1, \chi_2 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = x .$$

If we consider the action of $SU(2)$ on itself obtained by left translation, this identification allows us to consider $SU(2)$ as a subgroup of $SO(4)$. That is, for $x \in \mathbf{R}^4$ and $u \in SU(2)$ we let $ux = |x| \, u \, u_x$. The mapping $x \to ux$ so defined is easily seen to be a rotation.

The normalized Lebesgue surface measure on Σ_3, being invariant under rotation, is actually the Haar measure on $SU(2)$.[1]

In view of the Peter-Weyl theorem for $SU(2)$, a natural orthonormal basis for $L^2(SU(2)) = L^2(\Sigma_3)$ is obtained by considering the matrix entries of a complete system of irreducible representations.

We let $\mathcal{T}_u^{(k)}$ be the irreducible representation of $SU(2)$ realized on $\mathcal{P}^{(k)} = \mathcal{P}^{(k,2)}$, the space of homogeneous polynomials in $z = (z_1, z_2)$ of degree k, by letting

$$\mathcal{T}_u^{(k)} p(z) = p(u' z)$$

1) We have $\int_{SU(2)} f(u)\, du = \int_{\Sigma_3} f(u\xi)\, d\xi$.

As can be seen from (2.6) an orthonormal basis of $\mathscr{P}^{(k)}$ is given by

$$p_j(z) = \binom{k}{j}^{1/2} z_1^j z_2^{k-j} \qquad j = 0, 1, \ldots, k\,.$$

We define the matrix entries of $\mathscr{T}_u^{(k)}$ with respect to this basis by

$$p_j(u'z) = \sum_{l=0}^{k} \tau_{lj}^{(k)}(u)\, p_l(z)$$

and we obtain associated functions on $\mathbf{R}^4$, which we also denote by $\tau_{lj}^{(k)}$, if we define

$$\tau_{lj}^{(k)}(x) = |x|^k \tau_{lj}^{(k)}(u_x)$$

We then have the following result:

THEOREM (5.1). *The representations* $\mathscr{T}^{(k)}$ *form a complete system of irreducible representations of* SU (2). *The functions* $(k+1)\,\tau_{lj}^{(k)}(x)$ *constitute an orthonormal basis of* $\mathscr{H}_4^{(k)}$ *with respect to the inner product introduced in* (3.9).

Proof. The completeness of $\mathscr{T}^{(k)}$ follows from the second part of the theorem and the completeness of spherical harmonics on Σ_3.

For $|x| = 1$ $\tau_{lj}^{(k)}(x) = \tau_{lj}^{(k)}(u_x)$; thus, the orthogonality relations follow from the Peter-Weyl theorem.

The dimension of $\mathscr{H}_4^{(k)}$ is $(k+1)^2$ so that it remains to show that the functions $\tau_{lj}^{(k)}(x)$ are actually homogeneous harmonic polynomials of degree k.

We have by the binomial formula

$$\sum_{j=0}^{k} \frac{1}{\binom{k}{j}}\, p_j(z)\, \overline{p_j(w)} = (z.w)^k\,.$$

hence

$$(5.2) \qquad (u'z\cdot w)^k = \sum_{j=0}^{k} \frac{1}{\binom{k}{j}}\, p_j(u'z)\, \overline{p_j(w)} = \sum_{l,j=0}^{k} \frac{1}{\binom{k}{j}}\, \tau_{lj}^{(k)}(u)\, p_l(z)\, \overline{p_j(w)}$$

By the identification of $\mathbf{R}^4$ with $\mathbf{C}^2$ we have

$$(5.3) \qquad |x|\, u_x' z\cdot w = -i(-z_1\bar{w}_1\bar{\chi}_2 + z_2\bar{w}_1\bar{\chi}_1 + z_1\bar{w}_2\chi_1 + z_2\bar{w}_2\chi_2) =$$

$$= -i\left[(z_2\bar{w}_1 + z_1\bar{w}_2)\,x_1 - i(z_2\bar{w}_1 + z_1\bar{w}_2)\,x_2 + (z_2\bar{w}_2 - z_1\bar{w}_1)\,x_3 + {}\right.$$

$$\left. + i(z_2\bar{w}_2 + z_1\bar{w}_1)\,x_4\right] = \sum_{j=1}^{4} d_j x_j\,.$$

Thus from (5, 2), (5, 3)

$$(5.4) \qquad \sum_{l,j=0}^{k} \frac{1}{\binom{k}{j}} \tau_{lj}^{(k)}(x)\, p_l(z)\, \overline{p_j(w)} = \Big(\sum_{j=1}^{4} d_j x_j\Big)^k .$$

Each $\tau_{lj}^{(k)}(x)$ is a polynomial in $\mathscr{P}^k$. Moreover it is immediate that

$$\sum_{j=1}^{4} d_j^2 = 0 .$$

Thus

$$\Delta\Big(\sum_{j=1}^{4} d_j x_j\Big)^k = k(k-1)\Big(\sum_{j=1}^{4} d_j x_j\Big)^{k-2}\Big(\sum_{j=1}^{4} d_j^2\Big) = 0 .$$

This shows that

$$\tau_{lj}^{(k)}(x) \,\varepsilon\, \mathscr{H}_4^{(k)} .$$

We can now give an explicit formula for $\tau_{lj}^{(k)}(x)$:
Since

$$p_j(u_x' z) = \binom{k}{j}(-i)^k (z_2 \bar{\chi}_1 - z_1 \bar{\chi}_2)^j (z_1 \chi_1 + z_2 \chi_2)^{k-j} =$$

$$= \sum_{l=0}^{k} \tau_{lj}^{(k)}(x) \binom{k}{l} z_1^l z_2^{k-l} ,$$

letting $s = z_1/z_2$ we have

$$\binom{k}{j}(-i)^k (\bar{\chi}_1 - \bar{\chi}_2 s)^j (\bar{\chi}_1 s + \chi_2)^{k-j} = \sum_{l=0}^{k} \binom{k}{l} \tau_{lj}^{(k)}(x)\, s^l$$

Let

$$f(s) = \frac{\bar{x}_2}{|x|^2}(\chi_1 s + \chi_2), \quad 1 - f(s) = \frac{\chi_1}{|x|^2}(\bar{\chi}_1 - \bar{\chi}_2 s)$$

then

$$(-i)^k \binom{k}{j} [f(s)]^j [1 - f(s)]^{k-j} = \frac{\bar{\chi}_2^j \bar{\chi}_1^{k-j}}{|x|^{2k}} \sum_{l=0}^{k} \binom{k}{l} \tau_{ij}^k(x)\, s^l$$

and using Taylor's formula for the l^{th} coefficient in this sum we obtain the classical Jacobi polynomial expression (see Bateman [1] Vol. 2 pp. 254)

$$\tau_{lj}^{(k)}(x) = \frac{(-i)^k \binom{k}{j}}{\binom{k}{l}\, l\,!} |x|^{2k}\, \overline{(\chi_2^j \chi_1^{k-j})}^{\,-1} \frac{d^l}{dt^l}\big[t^j (1-t)^{k-j}\big] ,$$

where

$$t = \frac{\chi_2 \bar{x}_2}{|x|^2}.$$ [1]

§ 6. THE FOURIER TRANSFORM OF FUNCTIONS ON $\mathbf{R}^n$

We have shown that $L^2(\Sigma_{n-1})$ can be decomposed into a direct sum of mutually orthogonal subspaces (the spaces $\mathscr{H}_n^{(k)}$) that are invariant and irreducible under the action of rotations. There exists a corresponding decomposition of $L^2(\mathbf{R}^n)$ and the spaces making up this decomposition are intimately connected with the Fourier transform of functions of n real variables. In this section we shall construct these spaces and study the action of the Fourier transform restricted to them. We shall see that also in this situation the rotation group $SO(n)$ and its representations play a central role.

If f belongs to $L^1(\mathbf{R}^n)$ its *Fourier transform* $\hat{f}$ is defined by letting

$$(\mathscr{F} f)(y) = \hat{f}(y) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x.y} dx$$

for $y \in \mathbf{R}^n$.[1])

Perhaps the simplest class of functions that is invariant under the action of the Fourier transform is the collection of *radial functions.* We recall that these are the functions on $\mathbf{R}^n$ that depend only on $|x|$; equivalently, f is radial if $\rho_v f = f$ for all $v \in SO(n)$, where the operator ρ_v is defined by

$$(\rho_v f)(x) = f(v^{-1} x)$$

for all $x \in \mathbf{R}^n$. Since Lebesgue measure is invariant under the action of rotations and $v = v^*$ when $v \in SO(n)$,

$$\int_{\mathbf{R}^n} f(x) e^{-2\pi i x.v^{-1}y} dx = \int_{\mathbf{R}^n} f(x) e^{-2\pi i vx.y} dx = \int_{\mathbf{R}^n} f(v^{-1} x) e^{-2\pi i x.y} dx .$$

That is,

$$(\mathscr{F} \rho_v) f = (\rho_v \mathscr{F}) f \tag{6.1}$$

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[1]) It is not hard to use these results in order to obtain analogous results for $SO(3)$. We refer the reader to VILENKIN [11] for complete details.

[1]) When $f \varepsilon L^2(\mathbf{R}^n)$ the integral defining $\hat{f}$ is not defined in the Lebesgue sense. In this case, $\hat{f}$ is usually defined as the limit in the L^2 mean of the sequence $\hat{f}^k(y) = \int_{|x| \leq k} f(x) e^{-2\pi i x \cdot y} dx$. In order to avoid technical difficulties that arise from this definition we shall restrict our attention to integrable functions