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ANALYTIC SPACES

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CONTENTS

Chapter 1

ANALYTIC SPACES AND OPERATIONS ON THEM

We shall consider analytic spaces over the complex field **C** and sometimes over the real numbers **R**. Part of the results remain valid for spaces over arbitrary complete valuated fields but we shall restrict ourselves to the cases just mentioned.

1.1 Reduced analytic spaces.

To prepare for the general definition we shall first introduce reduced analytic spaces and their local models. Let U be an open set in $\mathbf{C}^n$ and V an analytic subset of U. The sheaf $\mathscr{I}$ on U of all germs of holomorphic functions vanishing on V is coherent by the Oka-Cartan theorem (for a proof, see e.g. Narasimhan [9, Theorem 5, p. 77]). The support of $\mathcal{O}_U/\mathscr{I}$ is V, and we shall denote by $\mathcal{O}_V$ the restriction of $\mathcal{O}_V/\mathscr{I}$ to V ($\mathcal{O}_U$ denotes the sheaf on U of germs of holomorphic functions). The *local models for reduced analytic spaces* shall be the pairs $(V, \mathcal{O}_V)$. Obviously we may consider $\mathcal{O}_V$ as a subsheaf of $\mathscr{C}_V$, the sheaf on V of germs of continuous functions.

Definition 1.1.1. A *reduced analytic space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space (not necessarily separated) and $\mathcal{O}_X$ is a sheaf of sub-$\mathbf{C}$-algebras of $\mathscr{C}_X$ which is locally isomorphic to a local model.

To be explicit, the last property means that every point $x \in X$ has a neighborhood U such that for some local model $(V, \mathcal{O}_V)$ there is a homeomorphism $\varphi : U \to V$ with the property that for $y \in U, f \in \mathscr{C}_{U,y}$ belongs to $\mathcal{O}_{U,y}$ if and only if $f = g \circ \varphi$ for some germ $g \in \mathcal{O}_{V,\varphi(y)}$.

As a common abuse of language we shall sometimes write X instead of $(X, \mathcal{O}_X)$.

Reduced analytic spaces need not be separated. Consider for example the disjoint union of two copies of $\mathbf{C}$, with all points except the origins identified. This topological space is in a natural way a reduced analytic space, indeed a complex manifold.

Reduced analytic spaces were introduced by Cartan-Serre (under the name of " analytic spaces ").

Definition 1.1.2. A *morphism*, or *holomorphic map* of one reduced analytic space $(X, \mathcal{O}_X)$ into another, $(Y, \mathcal{O}_Y)$, is a continuous map $\varphi : X \to Y$ such that $\varphi^* (\mathcal{O}_{Y,\varphi(x)}) \subset \mathcal{O}_X$ for all $x \in X$.

This definition, of course, gives us also the notion of isomorphism of reduced analytic spaces, which we have already used in a special case in Definition 1.1.1.

Example 1. If X, Y are complex manifolds, the morphisms of $(X,\mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$ are the holomorphic maps $X \to Y$ in the usual sense.

Example 2. The morphisms of $(X, \mathcal{O}_X)$ into $\mathbf{C}$, regarded as a reduced analytic space $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$, can be identified with the sections $\Gamma (X, \mathcal{O}_X)$.

Example 3. The morphisms of $(X,\mathcal{O}_X)$ into $\mathbf{C}^n$ can be identified with n-tuples of sections of $\mathcal{O}_X$, or, again, with sections of $\mathcal{O}_X^n$.

It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map $t \to (t^2, t^3)$ of $X = \mathbf{C}$ into the space Y of all pairs (x,y) satisfying $x^3 - y^2 = 0$. This is a bijective and bicontinuous morphism, but its inverse ψ is no morphism since $\psi^* f_0 \notin \mathcal{O}_{Y,0}$ if $f(t) = t$.

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider " Cartan's umbrella " which is the subset of $\mathbf{R}^3$ defined by the equation $z(x^2 + y^2) - x^3 = 0$. Its intersection with the plane $z = 1$ has an isolated double point at (0, 0, 1) and so it has a stick (the z-axis) joining the rest of the " umbrella " at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf $\mathcal{I}$ of germs of real-analytic functions vanishing on the umbrella were generated by sections $s_1, \ldots, s_n \in \Gamma(U, \mathcal{I})$ over some neighborhood U of the origin. Then, denoting by $f_1, \ldots, f_n$ the corresponding real-analytic functions in U, we find (using a complexification and the Nullstellensatz for principal ideals) that every f_j is a multiple of $z(x^2 + y^2) - x^3$ for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in $\mathcal{I}$ defined by the coordinate function x at a point $(0, 0, z)$, $z \neq 0$, cannot be a linear combination of $S_1, \ldots, S_n$ which is a contradiction.

1.2. Definition of general analytic spaces.

Let U be an open subset of $\mathbf{C}^n$ (or $\mathbf{R}^n$) and let $\mathcal{I}$ be an arbitrary coherent sheaf of ideals in $\mathcal{O}_U$, the sheaf on U of germs of holomorphic (or real-analytic) functions. Then $V = \operatorname{supp} \mathcal{O}_U/\mathcal{I}$ is an analytic subset of U. The restriction of $\mathcal{O}_U/\mathcal{I}$ to V will be denoted by $\mathcal{O}_V$. It is, in general, not a subsheaf of $\mathcal{C}_V$. The definition of a general analytic space will be based on *local models* $(V, \mathcal{O}_V)$ of the type just constructed. Note that a model $(V, \mathcal{O}_V)$ is of the previously considered reduced type if and only if $\mathcal{I}$ is the sheaf of *all* germs of holomorphic functions vanishing on V. In the general case the set V does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

Example 1. Let $U = \mathbf{C}$, $\mathcal{I}$ the sheaf of ideals generated by x^2. Here $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$ ($\mathbf{C}\{x\}$ denotes the space of converging power series in the variable x). Thus $\mathcal{O}_{V,0}$ is the space of " dual numbers " representable as $a + b\,\varepsilon$ where $a, b \in \mathbf{C}$ and $\varepsilon^2 = 0$, ε being the class of x. Evidently $\mathcal{O}_{V,0}$ cannot be a subring of the continuous functions on $\{0\}$. The only prime ideal of $\mathcal{O}_{V,0}$ is that generated by ε, hence the Krull dimension of $\mathcal{O}_{V,0}$ is 0. (Recall that the Krull dimension of a commutative ring A is the supremum of all numbers k such that there exists a strictly increasing chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_k$$

of prime ideals $\mathfrak{p}_j$.)

Example 2. Let V be the subspace of $\mathbf{C}^4$ defined by the requirement that $M(x) = \begin{pmatrix} x_1, & x_2 \\ x_3, & x_4 \end{pmatrix}$ be nilpotent. It can easily be seen that V can be defined by

$$\det M(x) = tr\, M(x) = 0 \tag{1}$$

and as well by

$$M(x)^2 = 0\,. \tag{2}$$

Let $\mathcal{I}$ and $\mathcal{I}'$ denote the sheaves of ideals defined by (1) and (2), respectively. Explicitly this means that $\mathcal{I}$ is generated by $x_1 + x_4$, $x_1 x_4 - x_2 x_3$ and $\mathcal{I}'$ by $x_1^2 + x_2 x_3$, $x_2(x_1 + x_4)$, $x_3(x_1 + x_4)$, $x_2 x_3 + x_4^2$. It can be seen easily that $\mathcal{I}' \subset \mathcal{I}$ but this inclusion is strict since the generators of $\mathcal{I}'$ are all of the second degree. Thus the two ideals provide two different structure sheaves on the same set V.

Example 3. Let us note here some less pleasant properties of real local models. Take, for example, $U = \mathbf{R}^2$, and let $\mathcal{I}$ be the sheaf of ideals generated by x^2+y^2. Then $V=\{0\}$ and $\mathcal{O}_{V,0} = \mathbf{R}\,\{x,y\}/(x^2 + y^2)$. Here $\{0\}$ and (x,y) are prime ideals so the Krull dimension of $\mathcal{O}_{V,0}$ is at least 1 (in fact it is 1) and therefore not equal to the geometric dimension of V as in the complex example above.

To give the definition of a general analytic space we first introduce that of a ringed space:

Definition 1.2.1. A **C**-*ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of local **C**-algebras. (This means that $\mathcal{O}_{X,x}$ are local algebras for $x \in X$ arbitrary; all algebras are assumed to be commutative and with units; furthermore $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is assumed to be isomorphic to **C** where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.)

Definition 1.2.2. A *morphism*

$$\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

of one **C**-ringed space into another is a pair $\varphi = (\varphi_0, \varphi^1)$ where $\varphi_0 : X \to Y$

is a continuous map, and $\varphi^1 : \varphi_0^* (\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of **C**-algebras (morphisms of algebras are always assumed to be unitary).

R-ringed spaces and their morphisms are of course defined similarly.

Let $f \in \Gamma(U, \mathcal{O}_X)$ be a section of a **C**-ringed space $(X, \mathcal{O}_X)$ over an open set $U \subset X$. We may then define the *value* $f(x)$ of f at a point $x \in U$ as $f_x \in \mathcal{O}_{X,x}$ taken modulo $\mathfrak{m}_x$. Since $\mathcal{O}_{X,x}/\mathfrak{M}_x \cong \mathbf{C}, f(x)$ is a complex number.

Example 4. The values $f(x)$ of f do not determine f completely. In the example

$$(\{0\},\ \mathbf{C}\{x\}/(x^2))$$

we considered earlier, the sections are given by dual numbers $a + b\,\varepsilon$, and since $\mathfrak{m}_0 = (\varepsilon)$, we get $f(0) = a$. Hence one has to consider also " higher order terms " to determine f.

If $\varphi : A \to B$ is a unitary homomorphism of local **C**-algebras it follows that $\varphi(\mathfrak{m}(A)) \subset \mathfrak{m}(B)$, $\mathfrak{m}(A)$ denoting the maximal ideal of A; in other words, the homomorphism is local. To see this, let us note that $\varphi^{-1}(\mathfrak{m}(B))$ is an ideal of A and that φ induces an injective (in fact bijective) map of $A/\varphi^{-1}(\mathfrak{m}(B))$ into $B/\mathfrak{m}(B) \cong \mathbf{C}$, hence $\varphi^{-1}(\mathfrak{m}(B))$ is either all of A or a maximal ideal in A, but the first possibility is ruled out by the condition $\varphi(1) = 1$. It therefore follows that $\varphi^{-1}(\mathfrak{m}(B)) = \mathfrak{m}(A)$, hence $\mathfrak{m}(B) \supset \varphi(\mathfrak{m}(A))$. A consequence of this is that a morphism $(\varphi_0, \varphi^1) : (X, \mathcal{O}_X) \to$ $\to (Y, \mathcal{O}_Y)$ of ringed spaces preserves the values of the sections, in symbols

$$(*) \qquad \varphi^1(f)(x) = f(\varphi_0(x)),$$

if $x \in X$ and f is a section of $\mathcal{O}_Y$ over some open set containing $\varphi_0(x)$. Thus φ^1 and φ_0 are related, but our example " the double point " shows that φ^1 is not in general determined by φ_0:

Example 5. Let X be the **C**-ringed space $(\{0\}, \mathbf{C}\{x\}/(x^2))$, and let $Y = \mathbf{C}^n$ regarded as a **C**-ringed space (with the sheaf $\mathcal{O}_{\mathbf{C}^n}$ of germs of holomorphic functions). Let (φ_0, φ^1) be a morphism of X into Y with $\varphi(0) = 0$, say. Then φ^1 is a homomorphism.

$$\varphi^1 : \mathbf{C}\{y_1, \ldots, y_n\} \to \mathbf{C}\{x\}/(x^2).$$

Let us express $\varphi^1(f)$ as $a(f) + \varepsilon b(f)$ (see the example[1]). Since the maximal ideal of $\mathbf{C}\{x\}/(x^2)$ is (ε), the value of $\varphi^1(f)$ is $a(f)$. From (*) it follows that

$$a(f) = \varphi^1(f)(0) = f(0) = \varphi_0^*(f).$$

Thus φ_0 determines the " zero order term " of $\varphi^1(f)(0)$. As to the properties of $b(f)$, it follows from the multiplication rule $\varepsilon^2 = 0$ that

$$b(fg) = f(0)\, b(g) + g(0)\, b(f),$$

hence that b is a tangent vector, or derivation, at $O \in \mathbf{C}^n$.

It is clear what the restriction of a ringed space $(X, \mathcal{O}_X)$ to an open subset U of X should mean: it is the ringed space $(U, \mathcal{O}_X | U)$. The following definition therefore makes sense.

Defintion 1.2.3. (Grothendieck [4]). A $\mathbf{C}$-*analytic space* is a $\mathbf{C}$-ringed space $(X, \mathcal{O}_X)$ where every point $x \in X$ has an open neighborhood U such that the restriction of $(X, \mathcal{O}_X)$ to U is isomorphic (in the sense of $\mathbf{C}$-ringed spaces) to a model (defined at the beginning of Section 1.2.). A morphism of analytic spaces is a morphism in the sense of ringed spaces.

We shall determine the morphisms of $(X, \mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$ in two important special cases, viz. when $(X, \mathcal{O}_X)$ is arbitrary and $(Y, \mathcal{O}_Y)$ is either $\mathbf{C}^n$ or defined by the vanishing of finitely many analytic functions in an open set in $\mathbf{C}^n$.

Proposition 1.2.4. The morphisms of a $\mathbf{C}$-analytic space $(X, \mathcal{O}_X)$ into $\mathbf{C}^n$ can be identified in a natural way with $\Gamma(X, \mathcal{O}_X)^n$ (or $\Gamma(X, \mathcal{O}_X^n)$).

Proof. Given a morphism $\varphi = (\varphi_0, \varphi^1)$ of $(X, \mathcal{O}_X)$ into $\mathbf{C}^n$ we shall construct an n-tuple $T\varphi = (f_1, \ldots, f_n)$ of sections of $\mathcal{O}_X$.

To define T we proceed as follows. Let $x \in X$. Recall that φ^1 maps $\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)}$ into $\mathcal{O}_{X,x}$. Define $(f_j)_x \in \mathcal{O}_{X,x}$ as the image under φ^1 of the germ at $\varphi_0(x)$ of the coordinate function y_j in $\mathbf{C}^n$. Somewhat less precisely, $f_j = \varphi^1(y_j)$. This defines $f_j \in \Gamma(X, \mathcal{O}_X)$ and hence T.

T is injective. For $T\varphi = T\psi$ means that

$$\mathcal{O}_{\mathbf{C}^n,\varphi_0(x)} \xrightarrow{\varphi^1} \mathcal{O}_{X,x}$$

and

$$\mathcal{O}_{\mathbf{C}^n,\psi_0(x)} \xrightarrow{\psi^1} \mathcal{O}_{X,x}$$

agree on the germs of the coordinate functions. Since in particular the *values* of the sections are preserved, i.e. φ^1 and ψ^1 are the identities modulo the respective maximal ideals, the *values* of the coordinates at $\varphi_0(x)$ and $\psi_0(x)$ must agree, hence $\varphi_0 = \psi_0$. Furthermore, since φ^1 and ψ^1 are homomorphisms, they agree on all polynomials. But the polynomials form a dense set in $\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)}$ and $\mathcal{O}_{X,x}$ is separated (for the Krull topology) in virtue of the Krull theorem (see Appendix). Finally φ^1 and ψ^1 are continuous maps since $\varphi^1(\mathfrak{m}(\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)})) \subset \mathfrak{m}(\mathcal{O}_{X,x})$. Now if two continuous maps from a topological space to a separated topological space coincide on a dense subset, then they are equal. Hence T is injective.

T is surjective. For if $(f_1, \ldots, f_n) \in \Gamma(X, \mathcal{O}_X)^n$ is given we first define $\varphi_0 : X \to \mathbf{C}^n$ by $\varphi_0(x) = (f_1(x), \ldots, f_n(x))$ (recall that $f(x)$ is the equivalence class of f_x modulo $\mathfrak{m}(\mathcal{O}_{X,x})$). Then we may define

$$\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)} \xrightarrow{\varphi^1} \mathcal{O}_{X,x}$$

first on the constants by the requirement that $\varphi^1(1) = 1$; then on the germs of the coordinates by putting $\varphi^1(y_j) = f_j$; next on the polynomials by the multiplicative property of homomorphisms and finally, by uniform continuity, in all of $\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)}$. (Note that we have again used the fact that $\mathcal{O}_{X,x}$ is separated in the last step).

Before the next proposition we introduce the notion of special model. A *special model* $(V, \mathcal{O}_V)$ is a model (see the beginning of this section) where the ideal $\mathcal{I}$ is generated by the components of a vector-valued analytic function $f : U \to F$ where U is open in $\mathbf{C}^n$ and F is a finite-dimensional complex linear space. Here V is the set of zeros of f and $\mathcal{O}_V$ is the restriction of $\mathcal{O}_U/\mathcal{I}$ to its own support.

Proposition 1.2.5. Let $(X, \mathcal{O}_X)$ be an arbitrary analytic space and $(Y, \mathcal{O}_Y)$ a special model defined by the vanishing of a vector-valued analytic function $g_0 : U \to G$. Then there is a bijection between the morphisms $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and those morphisms $\psi : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ which satisfy $g \circ \psi = 0$, where $g = (g_0, g^1) : (U, \mathcal{O}_U) \to (G, \mathcal{O}_G)$ is the morphism of analytic spaces defined by g_0.

The proof will be left as an exercise to the reader.

On the other hand, the morphisms $(X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ are obviously these morphisms $(X, \mathcal{O}_X) \to \mathbf{C}^n$ such that $\varphi_0(X) \subset U$; this fact, combined with propositions 1.2.4. and 1.2.5. gives the description of the morphisms: $(X, \mathcal{O}_X) \to$ (special model).

We end this section with the definition of analytic subspace. First we state

Definition. 1.2.6. An *analytic coherent sheaf* on an analytic space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$- modules such that every $x \in X$ has an open neighborhood U over which there exists an exact sequence

$$\mathcal{O}_X^q \mid U \to \mathcal{O}_X^p \mid U \to \mathcal{F} \mid U \to 0 .$$

Definition. 1.2.7. A *closed analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a ringed space $(Y, \mathcal{O}_Y)$ where $Y = \operatorname{supp}(\mathcal{O}_X/\mathcal{I})$ and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I} \mid Y$

for some coherent sheaf $\mathscr{I}$ of ideals of $\mathcal{O}_X$. An *open analytic subspace* of $(X, \mathcal{O}_X)$ is just a restriction $(U, \mathcal{O}_X \mid U)$, U open in X. An *analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a closed analytic subspace $(Y, \mathcal{O}_Y)$ of the open analytic subspace $(\complement \bar{Y} \cup Y, \mathcal{O}_{\complement \bar{Y} \cup Y})$ of $(X, \mathcal{O}_X)$, provided $\complement \bar{Y} \cup Y$ is indeed open in X, i.e. Y is locally closed in X.

Examples. The " single point " $(0, \mathbf{C})$ is an analytic subspace of the " double point " $(0, \mathbf{C}\{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. A " point " of an analytic space will always mean a single point embedded in $(X, \mathcal{O}_X)$ by means of a map $(0, \mathbf{C}) \to (X, \mathcal{O}_X)$.

1.3. Operations on analytic spaces.

In this section we shall write X for the analytic space $(X, \mathcal{O}_X)$.

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces X, X' is a triple (Z, π, π') where Z is an analytic space and $\pi : Z \to X$, $\pi' : Z \to X'$ are two morphisms with the following property:

Given any analytic space Y and any pair $f : Y \to X$, $f' : Y \to X'$ of morphisms there exists a unique morphism $g : Y \to Z$ such that $f = \pi \circ g$, $f' = \pi' \circ g$.

For example, the product of $\mathbf{C}^p$ and $\mathbf{C}^q$ is $\mathbf{C}^{p+q}$, according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of g clearly implies the uniqueness of the product (Z, π, π') up to isomorphism; we denote one such Z by $X \times X'$.

To prove that the product always exists, let us suppose first that X and X' are special models, i.e. X is defined by a triple (U, f, F) where U is open in $\mathbf{C}^n$, F is a finite-dimensional complex linear space, and $f : U \to F$ is an analytic map; similarly for X'. We claim that the special model Z defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \to X$, $\pi' : Z \to X'$ induced by the projections $U \times U' \to U$, $U \times U' \to U'$. Also, if $f: Y \to X$ and $f': Y \to X'$ are given, $g : Y \to Z$ is determined by

$$\begin{tikzcd}[row sep=small] & X \arrow[r] & U \arrow[dr] & \\ Y \arrow[ur, "f"] \arrow[dr, "f'"'] & & & U \times U' . \\ & X' \arrow[r] & U' \arrow[ur] & \end{tikzcd}$$

In the general case we take $X \times X'$ as the ringed space whose topological underlying space in the cartesian product of the underlying space of X and X', and whose structure sheaf is given locally by the product of local models for X and X'. (From the uniqueness "up to isomorphism" of the product results that these sheaves stick together in a well-determined way).

b) *Kernel of a double arrow.* If $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ is a double arrow, i.e. a pair of morphisms, a kernel X' of (u, v) is an analytic subspace of X such that the morphisms of an arbitrary analytic space Z into X' are exactly the morphisms h of $\dot{Z}$ into X such that $u \circ h = v \circ h$. In other words, if $i : X' \to X$ is the natural map of X' into X, the morphisms $h : Z \to X'$ satisfy $u \circ i \circ h = v \circ i \circ h$ and if a morphism $g: Z \to X$ satisfies $u \circ g = v \circ g$, then $g = i \circ h$ for some $h : Z \to X'$. To prove the existence of the kernel it suffices, again, to do this locally, i.e. for special models. If X is defined by (U, f, F) and Y by (V, g, G) we may (perhaps, after restricting U) extend u and v to maps $\bar{u}, \bar{v} : U \to E$ where E denotes the complex linear space of which V is an open subset. The kernel is then defined by the triple

$$\left(U, f \times (\bar{u} - \bar{v}), \quad F \times E\right).$$

It follows from the Proposition 1.2.5. that this special model satisfies the universal property of kernels.

Example 1. The kernel of $\mathbf{C} \underset{-t}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is the simple point $\{0\}$, t denoting the identity of $\mathbf{C}$.

Example 2. The kernel of $\mathbf{C} \underset{t+t^2}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is $\{0\}$ counted as a double point.

c) *Fiber product.* If $u : X \to S$ and $v : Y \to S$ are given morphisms of analytic spaces, the fiber product $X \times_S Y$ of X and Y over S is the kernel of the double arrow

$$X \times Y \underset{v \circ \pi'}{\overset{u \circ \pi}{\rightrightarrows}} S$$

where $\pi : X \times Y \to X$ and $\pi' : X \times Y \to Y$ are the maps defined by the product. Note that when S is a simple point, $X \times_S Y = X \times Y$.

One may also introduce the category of analytic spaces over S. Its objects are morphisms $u : X \to S$ of an analytic space X onto S and its morphisms are morphisms $f : X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_u\searrow \quad \swarrow_v & \\ & S & \end{array}$$

is commutative. The product in this category, i.e. the object satisfying the universal property given above for the product $X \times Y$, is then exactly the fiber product $X \times_s Y$. If S is a point, we have the category of analytic spaces.

Example 3. If U and V are open subspaces of an analytic space X, the open subspace $U \cap V$ is isomorphic to $U \times_X V$. We may thus define, in general, the intersection of two analytic subspaces $X' \to X$ and $X'' \to X$ of X to be the fiber product $X' \times_X X''$.

Example 4. If $\varphi : Y \to X$ is a morphism of analytic spaces and $a \in X$ a point, i.e. a map $a : (0, \mathbf{C}) \to X$ we may consider the space $Y(a) = Y \times_X a$. It is natural to call this the inverse image of a under φ and to denote it by $\varphi^{-1}(a)$; its underlying space is exactly $\varphi_0^{-1}(a)$.

If $\varphi_0(b) = a$, then $\mathcal{O}_{Y(a),b}$ is $\mathcal{O}_{Y,b}$ taken modulo the image under $\varphi^1 : \mathcal{O}_{X,a} \to \mathcal{O}_{Y,b}$ of the maximal ideal in $\mathcal{O}_{X,a}$.

Example 5. The pull-back of a linear bundle E over X by a map $Y \to X$ is exactly $Y \times_X E$.

1.4. *Relations between reduced and non-reduced spaces.*

We shall first characterize those analytic spaces which are reduced.

Proposition 1.4.1. A analytic space $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent element for x arbitrary in X.

Proof. The necessity of the condition is obvious for $\mathcal{O}_X$ can be considered as a submodule of $\mathcal{C}_X$ if $(X, \mathcal{O}_X)$ is reduced.

Conversely, if $\mathcal{O}_{X,x}$ has no nilpotent elements, we shall prove that in any local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$, a germ g at $a \in V$ which vanishes on V belongs to the ideal $\mathcal{I}$ defining $\mathcal{O}_V$. The Nullstellensatz implies that $g^k \in \mathcal{I}_a$ if k is large enough. But it is then clear that $g \in \mathcal{I}_a$ if $\mathcal{O}_{V,a}/\mathcal{I}_a$ is free from nilpolent elements.

Given an analytic space $(X, \mathcal{O}_X)$ we can associate to it a reduced space in the following way. Let $\mathcal{N}_x$ be the ideal in $\mathcal{O}_{X,x}$ consisting of all nilpotent elements (the nil-radical of 0). Then $\mathcal{N} = U\mathcal{N}_x$ is a coherent sheaf by the Oka-Cartan theorem, for in a local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$ we have $\mathcal{N}_X = (\mathcal{I}'/\mathcal{I})_X$ where $\mathcal{I}'$ is the sheaf of germs vanishing on V and $\mathcal{I}$ the sheaf of ideals defining $\mathcal{O}_V$. The sheaf $\mathcal{I}'$ is coherent by the Oka-Cartan then, and $\mathcal{I}$ by assumption, hence $\mathcal{I}'/\mathcal{I}$ is coherent. Now define $(X_{red}, \mathcal{O}_{X red})$ by taking X_{red} equal to X as a topological space, and $\mathcal{O}_{X red} = \mathcal{O}_X/\mathcal{N}$.

For a systematic treatment of reduced analytic spaces we refer to Narasimhan [9]. We remark here that for non-reduced spaces, the decomposition into irreducible components has no meaning, even at a point.

Example. Consider the analytic subspace X of $\mathbf{C}^2$ defined by the ideal $\mathcal{I}$ generated by $x_1 x_2$ and x_2^2. It is clear that $\mathcal{I}_X = (x_2)$ if $x_1, \neq 0$, hence X is locally the one-dimensional manifold $x_2 = 0$ outside the origin. However, $\mathcal{I} = (x_2) \cap (x_1, x_2^2)$ which is strictly contained in (x_2) at the origin so the origin cannot be an ordinary point, in particular X is not an analytic subspace of the manifold $x_2 = 0$. To illustrate this further, let $\pi : X \to \mathbf{C}$ be the projection of X into $\mathbf{C}$ defined by $(x_1, x_2) \to x_1$. We shall calculate the fibers $\pi^{-1}(a) = X \times_{\mathbf{C}} \{a\}$ of this map for an arbitrary point $a \in \mathbf{C}$.

To do this, we use the characterisation of $\mathcal{O}_{\pi^{-1}(a),b}$ given in § 1.3, example 4: if $a (= x_1) \neq 0$, and $b = (a,0)$ we find immediately $\mathcal{O}_{\pi^{-1}(a),b} = \mathbf{C}$ hence $\pi^{-1}(a)$ is a simple point. But, if $a = 0$, $b = (0,0)$ we find $\mathcal{O}_{\pi^{-1}(a),b} = \mathbf{C}\{x_1, x_2\}/(x_1, x_2^2) \simeq \mathbf{C}\{x_2\}/(x_2^2)$; hence $\pi^{-1}(0)$ is a double point.

Chapter 2.

DIFFERENTIAL CALCULUS ON ANALYTIC SPACES

Very little is known yet about differential operators on spaces with singularities. We shall just give the main definitions here. Let us first consider differential operators in the regular case, i.e. on manifolds. One then usually introduces, for each point a on a complex manifold X, the vector space $\mathcal{O}_{X,a}/\mathfrak{m}_a^{k+1}$, the jets of order k at a. Here $\mathfrak{m}_a$ denotes, as usual, the maximal ideal in $\mathcal{O}_{X,a}$. The jets of order k form, in a natural way, an analytic bundle J^k. A differential operator is then by definition a morphism of J^k into the trivial bundle $X \times \mathbf{C}$. Differential operators from bundles to bundles are defined similarly.

This definition is not suitable for generalization to analytic spaces (the collection of vector spaces $\mathcal{O}_{X,a}/\mathfrak{m}_a^{k+1}$ would not define a bundle over X). However, as noted by Grothendieck [4], if we consider, instead of the bundle J^k, the sheaf of sections of it, we can generalize to any analytic space X the definition above in the following way:

Let Δ be the diagonal in $X \times X$ regarded as a closed analytic subspace, i.e. the sheaf $\mathscr{I}$ of ideal defining Δ is that generated by all germs of the form $\pi_1{}^* f - \pi_2{}^* f$ where f is a germ on X, $\pi_j : X \times X \to X$ being the projections. Similarly $\Delta^{(k)}$ denotes the analytic subspace of X^2 with sheaf of ideals $\mathscr{I}^{k+1}$ ($\Delta^{(k)}$ is not reduced for $k > 1$ even if X is). The structure sheaf $\mathcal{O}_{\Delta(k)}$ on $\Delta^{(k)}$ is moved down to X by π_1 ; its direct image will be denoted by $\pi_{1*}\, \mathcal{O}_{\Delta(k)}$, a sheaf on X. It is made into an $\mathcal{O}_X$-module by the map $\mathcal{O}_X \to \pi_{1*}\, \mathcal{O}_{\Delta(k)}$ defined by $\pi_1{}^* : \mathcal{O}_{X,x} \to \mathcal{O}_{X\times X;(x,\alpha)}$.

Definition. (Grothendieck). A linear differential operator of order $\leqslant k$ is a morphism $\pi_{1*}\, \mathcal{O}_{\Delta(k)} \to \mathcal{O}_X$, both sheaves being considered as $\mathcal{O}_X$-modules.

Let us see how this definition connects with the usual one in case X is a manifold.

Differential operators in $\mathbf{C}^n$. Let U be open in $\mathbf{C}^n$ (or a coordinate patch on a manifold). Then a differential operator in the usual sense in U is a map $Q : \mathcal{O}_U \to \mathcal{O}_U$ of the form

$$f \to \sum_{|j|\leq k} a_j D^j f$$

where a_j are analytic functions in U and

$$D^j f = \frac{\partial^{j1}}{\partial x_1^{j1}} \cdots \frac{\partial^{jn}}{\partial x_n^{jn}} f\,.$$

Clearly Q is $\mathbf{C}$-linear and continuous. Consider the map φ ; $\mathcal{O}_U \to \pi_{1*}\, \mathcal{O}_{\Delta(k)}$ defined as the composition of $\pi_2{}^* : \mathcal{O}_U \to \mathcal{O}_{\Delta(k)}$ and the natural map $\to \pi_{1*}\, \mathcal{O}_{\Delta(k)}$. In somewhat sloppy notation,

$$f(x) \xrightarrow{\varphi} \sum_{|j|\leq k} D^j f(x)\,(y-x)^j/j\,!\,,$$

where $j! = j_1! \dots j_n!$. Now if $P : \pi_{1*}\, \mathcal{O}_{\Delta(k)} \to \mathcal{O}_U$ is a differential operator in the sense of the definition just made, we get a differential operator in the elementary sense by putting $Q = P \circ \varphi$. Here $a_j(x) = P((y-x)^j/j!)$ are sections of $\mathcal{O}_U$ over all of U, for P maps sections of $\Gamma(U, \pi_{1*}\mathcal{O}_{\Delta(k)})$ onto sections of $\Gamma(U, \mathcal{O}_U)$.

Conversely, if Q is given, P can be constructed from the requirement $P((y-x)^j) = j!\, a_j$, for the germs $(y-x)^j$, $|j| \leqslant k$, generate $\pi_{1*}\, \mathcal{O}_{\Delta(k)}$ as an $\mathcal{O}_U$-module. By this procedure every linear differential operator $\in \mathrm{Hom}_{\mathcal{O}_U}(\pi_{1*}\, \mathcal{O}_{\Delta(k)}, \mathcal{O}_U)$ defines an element of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{O}_U, \mathcal{O}_U)$; hence every germ $\in$

$Hom_{\mathcal{O}_U}(\pi_{1*}\,\mathcal{O}_{\Delta(k)}, \mathcal{O}_V)$ of a differential operator determines an element of $Hom_{\mathbf{C}}(\mathcal{O}_U, \mathcal{O}_U)$.

Lifting of differential operators, in models. We shall also describe in a more concrete way the differential operators in a model $(X, \mathcal{O}_X)$ where $X = \operatorname{supp} \mathcal{O}_U/\mathcal{J}$, $\mathcal{O}_X = (\mathcal{O}_U/\mathcal{J}) \mid X$, U being a domain of holomorphy in $\mathbf{C}^n$, $\mathcal{J}$ a coherent sheaf of ideals in $\mathcal{O}_U$. We claim that the differential operators in X correspond to those differential operators in the elementary sense in U which map $\mathcal{J}$ into $\mathcal{J}$, taken modulo those which map $\mathcal{O}_U$ into $\mathcal{J}$. Consider the following diagram where all arrows except Q, P, P_1 are ring homomorphisms:

$$\begin{tikzcd}
\mathcal{O}_U \arrow[d, "\pi_2^*"] \arrow[rr, "Q"] & & \mathcal{O}_U \\
\mathcal{O}_{U\times U}|_{\Delta U} \arrow[r] \arrow[d] & \pi_{1*}\,\mathcal{O}_{\Delta^{(k)}_U} \arrow[r, "P"] \arrow[d, "\Psi"] & \mathcal{O}_U \arrow[d, "\psi"] \\
\mathcal{O}_{X\times X}|_{\Delta X} \arrow[r] & \pi_{1*}\,\mathcal{O}_{\Delta^{(k)}_X} \arrow[r, "P_1"] & \mathcal{O}_X
\end{tikzcd}$$

First, if P_1 is a given differential operator in X we may construct an operator P in U (and hence an operator Q in the elementary sense) as follows. To give P it is sufficient to give the sections a_j, $|j| \leqslant k$, onto which $(x-y)^j/j\,!$ are to be mapped (see the previous section). The image in $\mathcal{O}_X$ of the sections $(x-y)^j/j\,!$ by $P_1 \circ \psi$ are certain sections b_j. In view of Theorem B of Cartan these can be lifted to sections a_j of $\mathcal{O}_U$ over U. The ambiguity in constructing a_j corresponds exactly to an operator mapping $\pi_{1*}\,\mathcal{O}_{\Delta(k)_U}$ into $\mathcal{J}$. Let us also note that the corresponding operator $Q : \mathcal{O}_U \to \mathcal{O}_U$ has the claimed property that $Q(\mathcal{J}) \subset \mathcal{J}$. In fact, if $f \in \mathcal{J}$, the image of f down in $\mathcal{O}_{X\times X} \mid_{\Delta X}$ is already zero, a fortiori its image in $\mathcal{O}_X$ is zero. Since the diagram is commutative it follows that the image of f by Q is in $\mathcal{J}$.

Conversely, suppose that Q is given, $Q(\mathcal{J}) \subset \mathcal{J}$, and that P is constructed from Q as before. We shall then find P_1 to make the diagram commutative. We clearly have to define $P_1\, g$ by first lifting $g \in \pi_{1*}\,\mathcal{O}_{\Delta^{(k)}_X}$ to $\pi_{1*}\,\mathcal{O}_{\Delta^{(k)}_U}$, then take $\psi \circ P$ of the element thus obtained. To see that this definition is allowed we have to see that $\ker \psi \subset \ker(\psi \circ P)$. However, it is clear that $\ker \psi$ is generated by the images in $\pi_{1*}\,\mathcal{O}_{\Delta^{(k)}_U}$ of $\pi_1{}^*\,\mathcal{J}$ and $\pi_2{}^*\,\mathcal{J}$. Now if $f \in \pi_1{}^*\,\mathcal{J}$, its image in $\mathcal{O}_U$ is $a_0(x) f(x)$ which belongs to $\mathcal{J} = \ker \psi$. On the other hand, if $f \in \pi_2{}^*\,\mathcal{J}$, its image in $\mathcal{O}_U$ is contained in $\mathcal{J}$ by our assumption on Q. This proves that P_1 is well-defined.

Example 1. Let us determine all differential operators on the double point $(0, \mathbf{C}\{x\}/(x^2))$. By the principle of lifting differential operators we shall therefore decide when a differential operator in **C**,

$$Q = \sum_{o}^{k} a_j(x) \frac{\partial^j}{\partial x^j}$$

maps (x^2) into (x^2). First we reduce the coefficients modulo (x^2) so that

$$Q = \sum_{o}^{k} (b_j + c_j x) \frac{\partial^j}{\partial x^j}$$

where b_j, c_j are complex numbers. It is clearly necessary and sufficient that

$$Q\left(\frac{x^k}{k!}\right) \equiv 0 \ (\mathrm{mod}\ x^2)\,, \qquad k \geqslant 2\,.$$

This is equivalent to

$$b_k = c_k + b_{k-1} = 0\,, \qquad k \geqslant 2\,,$$

hence the differential operators are precisely

$$b_0 + c_0\, x + (b_1 + c_1\, x) \frac{\partial}{\partial x} - b_1\, x \frac{\partial^2}{\partial x^2}\,.$$

This gives a space of dimension 4 on **C**, with the following basis

$$Q_1 = \text{identity}\,; \qquad Q_2 = x\,; \qquad Q_3 = x \frac{\partial}{\partial x}\,; \qquad Q_4 = \frac{\partial}{\partial x} - x \frac{\partial^2}{\partial x^2}$$

(this last being of order two!). Note that all the **C**-linear maps of the space of dual numbers into itself are given by differential operators.

We define the composition of differential operators as in the non singular case, by the composition of the corresponding elements of Hom $(\mathcal{O}_X, \mathcal{O}_X)$ (we leave the details to the reader) ; if P has order $\leqslant p$ and Q order $\leqslant q$, the $P\ Q$ has order $\leqslant p + q$. Denote by $\mathcal{D}_{X,x}$ the space of germs of differential operators of any order on X at the point x ; with this operation, $\mathcal{D}_{X,x}$ is a (non-commutative) ring.

Very little is known on these rings, except in the non-singular case. For instance:

1) Are they " finitely generated " in the sense that there would exist $D_1, \ldots$, $D_k \in \mathcal{D}_{X,x}$ such that any $D \in \mathcal{D}_{X,x}$ could be written as $D = \sum f_{i_1 \ldots i_p} D_{i_1} \ldots D_{i_p}$ $(f_{i_1}, \ldots, {}_{i_p} \in \mathcal{O}_{X,x}\ ;\ i_1, \ldots, i_p = 1, \ldots, k)$?
2) Are they left or right noetherians ? (In the non-singular case, to prove this result, it suffices to introduce the filtration defined by the order and

to note that the associated graded ring is a ring of polynomials on $\mathcal{O}_{X,x}$, and therefore is noetherian).

The differential. If f is a holomorphic germ on an analytic space $(X, \mathcal{O}_X)$ we define its differential $d f$ as the image by $\pi_1{}^*$ of the germ $\pi_2{}^* f - \pi_1{}^* f$ in $\mathcal{O}_{\Delta(1)}$; $\pi_j : \Delta^{(1)} \to X$ being the two natural projections induced by the projections $X^2 \to X$. Obviously $\pi_2{}^* f - \pi_1{}^* f$ vanishes on the diagonal, i.e. it belongs to the sheaf $\tilde{\Omega}_X$ of ideals of germs in $\mathcal{O}_{\Delta(1)}$ which have restriction zero to $\Delta = \Delta^{(0)}$. We call $\Omega_X = \pi_{1*} \tilde{\Omega}_X$ the *sheaf of (first order) differentials* on X. Clearly we have a natural isomorphism

$$\pi_{1*} \mathcal{O}_{\Delta(1)} \cong \mathcal{O}_X \oplus \Omega_X .$$

In a local model $(V, \mathcal{O}_V)$, V an analytic subset of $U \subset \mathbf{C}^n$, U open, we can also introduce the sheaf of differentials as follows. Let Ω_U denote the sheaf of germs of differential 1-forms on U.

Suppose that the sheaf of ideals defining $\mathcal{O}_V$ is generated by $f_1, \ldots, f_p$ Then Ω_U modulo the subsheaf $(f_1, \ldots, f_p)\, \Omega_U + \mathcal{O}_U\, (d f_1, \ldots, d f_p)$ defines a sheaf with support equal to V which coincides with the sheaf of differentials on V as defined above.

Vector fields. A *germ ξ of a vector field* at a point x of an analytic space is the same as a first order homogeneous (i.e. $\xi\,(1) = 0$) differential operator at x. In other words, ξ is defined as an $\mathcal{O}_{X,x}$-linear map of the germs of differential 1-forms at x into $\mathcal{O}_{X,x}$. A *vector field* on X is, of course, a section of the sheaf of germs of vector fields so defined.

Example 2. Consider the analytic subspace of $\mathbf{C}^2$ defined by the ideal $(x^3 - y^2)$. Here all vector fields are linear combinations of the equivalence classes of $2\, x\, \partial/\partial\, x + 3\, y\, \partial/\partial\, y$ and $2\, y\, \partial/\partial\, x + 3\, x^2\, \partial/\partial y$. In particular, all vector fields vanish at the origin. To see this, it is only necessary to observe that a differential operator $a\,(x, y)\, \partial/\partial\, x + b\,(x, y)\, \partial/\partial\, y$ must give a multiple of $x^3 - y^2$ when applied to $x^3 - y^2$ if it shall operate on the ring $\mathbf{C}\,\{x, y\} /(x^3 - y^2)$. Hence it must satisfy $3\, y\, a\,(x, y) - 2\, x\, b\,(x, y) \equiv 0 \pmod{x^3 - y^2}$. The space of these operators is spanned by the two just given, modulo $x^3 - y^2$.

If ξ is a vector field on X, one can define, as in the non-singular case, the " local group of automorphisms " $\exp\,(t\, \xi)$: it suffices to consider the case of a local model, when X is a closed subspace of U open, $\subset \mathbf{C}^n$, and ξ is the restriction of a vector field $\tilde{\xi}$ on U, and to note that $\exp\,(t\, \tilde{\xi})$ leaves X invariant. Suppose f. i. that X has an isolated singular point at x : then exp $(t\,\xi)$ must leave x invariant, and therefore ξ must vanish at x (this was the case in the preceding example).

The Zariski tangent space. The Zariski tangent space at a point x of an analytic space X is the dual over $\mathbf{C}$ of $\mathfrak{M}_x/\mathfrak{M}_x^2$; here $\mathfrak{M}_x$ denotes as usual the maximal ideal of $\mathcal{O}_{X,x}$. If X is defined by the ideal $\mathcal{I} \subset \mathcal{O}_U$, U an open set in $\mathbf{C}^n$, the tangent space may be identified with the linear variety defined by the linear parts of all germs $\in \mathcal{I}_x$.

The Zariski tangent space of X_{red} may be strictly contained in that of X. For instance, if X is a double point, $\mathfrak{M}_x/\mathfrak{M}_x{}^2$ has dimension 1 over $\mathbf{C}$ whereas $\mathfrak{M}_x/\mathfrak{M}_x{}^2 = \{0\}$ for X_{red}, the corresponding simple point.

The tangent cone. The tangent cone at a point x of a local model $(X, \mathcal{O}_X)$ is the algebraic variety (with nilpotents, in general) defined by the ideal generated by the first non-vanishing homogeneous parts of the elements in $\mathcal{I}_x$, $\mathcal{I}$ being the ideal defining X. Since the Zariski tangent space is defined, in the local model, by the ideal spanned by the first-degree parts of the elements of $\mathcal{I}_x$ it is clear that it contains, and in general strictly, the tangent cone. If ξ is a vector field, $\xi\,(x)$ belongs to the reduced tangent cone at x, but since the possible values of $\xi\,(x)$ form a linear space, it is in general not equal to the whole cone.

Example 3. Let, again, X be the analytic subspace of $\mathbf{C}^2$ defined by the ideal $(x^3 - y^2)$. Then, as noted before, $\xi\,(x) = 0$ for all possible vector fields; the tangent cone is the algebraic variety defined by the ideal (y^2), and the reduced tangent cone is the variety $y = 0$; finally, the Zariski tangent space is the whole space $\mathbf{C}^2$, for $x^3 - y^2$ contains no linear terms.

Chapter 3.

FINITE MORPHISMS

3. 1. *Local theory.*

As elsewhere in these notes, we denote by $\mathbf{C}\{x_1, ..., x_n\}$ the ring of convergent power series in n variables $x_1, ..., x_n$ First, we recall the so-called " Weierstrass preparation theorem ".

Theorem 3.1.1. (Späh, Rückert). Given $\Phi \in \mathbf{C}\{x_1, ..., x_n\}$, with $\Phi\,(0, ..., 0, x_n) = x_n{}^p +$ (higher order terms), any $f \in \mathbf{C}\{x_1, ..., x_n\}$ can be written

$$f = \Phi Q + \sum_{i=0}^{p-1} x_n^i a_i$$

with $Q \in \mathbf{C}\{x_1, ..., x_n\}$, $a_i \in \mathbf{C}\{x_i, ..., x_{n-1}\}$
This representation is unique.

Corollary 3.1.2. (Weierstrass). Given Φ as in the preceding theorem, there exist $u \in \mathbf{C}\{x_1, ..., x_n\}$, with $u(0) \neq 0$ and $a_i \in \mathbf{C}\{x_1, ..., x_{n-1}\}$, with $a_i(0) = 0$ such that

$$\Phi u = x_n^p + \sum_{i=0}^{p-1} a_i x_n^i$$

u and (a_i) are unique.

The corollary results easily from the theorem, when applied to $f = x_n^p$. For the proof of theorem 3.1.1., see e.g. [5] or [9]. We recall also that theorem 3.1.1. implies the facts that $\mathbf{C}\{x_1, ..., x_n\}$ is noetherian, and is a unique factorisation domain.

Definition 3.1.3. An analytic algebra (we shall say also " analytic ring ") is a quotient $\mathbf{C}\{x_1, ..., x_n\}/\mathscr{I}$, where $\mathscr{I}$ is a non trivial ideal (*i.e.* $\mathscr{I} \neq \mathbf{C}\{x_1, ..., x_n\}$). An analytic algebra A is clearly a local $\mathbf{C}$-algebra ; we denote by $\mathfrak{M}(A)$ its maximal ideal; we have $A/\mathfrak{M}(A) \simeq \mathbf{C}$.

An analytic algebra, being a quotient of a noetherian ring, is a noetherian ring, and therefore is separated in the Krull topology (see appendix).

If A and B are two analytic algebras, and $f: A \to B$ a homomorphism (with $f(1) = 1$), we recall that f is automatically local and therefore continuous in the Krull topology (see § 1.2.). If E is a B-module (unitary), then the map $A \times E \to E$ defined by $(a, e) \to f(a)\, e$ makes E an A-module ; for simplicity, we write $f(a)\, e = a\, e$.

We can now state the preparation theorem, in the general form :

Theorem 3.1.3. Let A and B be analytic algebres, f a homomorphism $A \to B$, and E a finite B-module. Then E is finite over A if and only if $E/\mathfrak{M}(A)E$ is finite over $A/\mathfrak{M}(A) \simeq \mathbf{C}$ (by " finite over A " we mean " finitely generated as an A-module ").

This theorem can be precised as follows :

Corollary 3.1.4. Given A, B, f, E as above, suppose that the images of $e_1, ..., e_p$ in $E/\mathfrak{M}(A)E$ generate that module over $\mathbf{C}$; then $e_1, ..., e_p$ generate E over A.

Proof of the corollary, admitting the theorem. Let F be the sub-A-module of E spanned by $e_1, \ldots, e_p$; then, by hypothesis, we have $E = F + \mathfrak{M}(A)E$. On the other hand, using the theorem, we know that E is finite over A; we can therefore apply Nakayama's lemme (see Appendix), which proves the corollary.

The existence part of theorem 3.1.1. is a special case of the preceding result. For, we take $A = \mathbf{C}\{x_1, \ldots, x_{n-1}\}$, $B = \mathbf{C}\{x_1, \ldots, x_n\}$ and f the natural injection (or, in a more sophisticated language, $f = \pi^*$ where π is the projection $\mathbf{C}^n \to \mathbf{C}^{n-1}$ which " forgets the last coordinate ") ; choose now Φ as in theorem 3.1.1. and $E = B/(\Phi)$. Then $E/\mathfrak{M}(A)E$ is isomorphic to $\mathbf{C}\{x_n\}/(\Phi(0, \ldots, 0, x_n)) = \mathbf{C}\{x_n\}/(x_n{}^p)$, which is generated over $\mathbf{C}$ by the classes of $1, x_n, \ldots, x_n{}^{p-1}$. Therefore, the corollory 3.1.4. shows that the classes of $1, x_n, \ldots, x_n{}^{p-1}$ in E generates E over A, which is the existence part of theorem 3.1.1.

A direct proof of theorem 3.1.3. in a slightly less general case ($E = B$) can be found in [6] (the general case could be easily deduced of it). We shall follow here another method, used by Mather [7] in the $\mathbf{C}^\infty$-case, and deduce theorem 3.1.3. from theorem 3.1.1. We proceed in three steps.

Step 1. $A = \mathbf{C}\{x_1, \ldots, x_{n-1}\}$, $B = \mathbf{C}\{x_1, \ldots, x_n\}$, $f = \pi^*$, the natural injection $A \to B$. As in the theorem, E is a finite B-module such that $E/\mathfrak{M}(A)E$ is finite over $\mathbf{C}$.

We first prove the existence of a finite number of elements $e_1, \ldots, e_p$ in E such that any $e \in E$ can be written $e = \Sigma b_i e_i$, with $b_i \in f(A) + \mathfrak{M}(A)B$. To this end, let $\varepsilon_1, \ldots, \varepsilon_q$ generate E over B, and let $\eta_1, \ldots, \eta_r$ be members of E such that their classes $\bar{\eta}_1, \ldots, \bar{\eta}_r$ modulo $\mathfrak{M}(A)E$ generate $E/\mathfrak{M}(A)E$ over $\mathbf{C}$. Thus, for any $e \in E$, we have, for suitables $\gamma_i \in \mathbf{C}$

$$e - \Sigma\gamma_i \eta_i \in \mathfrak{M}(A)E$$

and therefore

$$e - \Sigma\gamma_i \eta_i = \Sigma b_j \varepsilon_j, \quad b_j \in \mathfrak{M}(A)B$$

and it suffices to take $p = q + r$, $(e_1, \ldots, e_p) = (\eta_1, \ldots, \eta_r, \varepsilon_1, \ldots, \varepsilon_q)$

Therefore, for $1 \leqslant i \leqslant p$, we have

$$x_n e_i = \sum_j v_{ij} e_j, \quad v_{ij} \in f(A) + \mathfrak{M}(A)B$$

(in other words, $v_{ij}(0, \ldots, 0, x_n)$ is a constant). If we put $\Phi = \det(x_n \delta_{ij} - v_{ij})$, we have $\Phi e_i = 0$ $i = 1, \ldots, p$, then $\Phi E = 0$. Therefore E is a module over $B/(\Phi)$, generated e.g. by $e_1, \ldots, e_p$. But $\Phi(0, \ldots, 0, x_n)$ is a monic polynomial of degree p, and therefore is not identically zero ; by theorem

3.1.1., $B/(\Phi)$ is finite over A, and is generated by $1, x_n, \ldots, x_n^k$ for some $k \leqslant p - 1$. Then E is finite over A.

Step 2. We suppose that A and B are regular analytic rings:

$$A = \mathbf{C}\{x_1, \ldots, x_n\}, \qquad B = \mathbf{C}\{y_1, \ldots, y_m\}$$

and let f be any homomorphism $A \to B$.

We factorise f in the following way

$$\begin{tikzcd}[column sep=small] & C = \mathbf{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\} \arrow[dr, "j"] & \\ A = \mathbf{C}\{x_1, \ldots, x_n\} \arrow[ur, "i"] \arrow[rr, "f"] & & B = \mathbf{C}\{y_1, \ldots, y_m\} \end{tikzcd}$$

where i is the natural injection, and $\tilde{f}$ is "the map into the graph" defined by

$$\tilde{f}(x_i) = f(x_i), \qquad \tilde{f}(y_j) = y_j$$

By our hypothesis, E is finite over B; then, $\tilde{f}$ being surjective, E is finite over C; the problem is now reduced to one similar to the first case, except that the number of additional variables is m instead of 1. The proof follows by repeated use of step 1.

Step 3. General case. We have now $A = \mathbf{C}\{x_1, \ldots, x_n\}/\mathscr{I}$, $B = \mathbf{C}\{y_1, \ldots, y_m\}/\mathscr{J}$; and E is a finite B-module such that $E/\mathfrak{M}(A)E$ is finite over $\mathbf{C}$.

First, we put $A' = \mathbf{C}\{x_1, \ldots, x_n\}$ and we denote by f' $A \to B$ the composition of f and the natural projection $A' \to A$; it is clear that $E/\mathfrak{M}(A')E \simeq E/\mathfrak{M}(A)E$; therefore, we can replace A by A'.

Now, putting $B' = \mathbf{C}\{y_1, \ldots, y_m\}$ and π the natural projection $B' \to B$, we claim that there exists a commutative diagram of homomorphisms

$$\begin{tikzcd} & B' \arrow[d, "\pi"] \\ A' \arrow[ur, "\tilde{f}"] \arrow[r, "f'"'] & B \end{tikzcd}$$

For, let $\varphi_1, \ldots, \varphi_n \in B'$ be liftings of $f'(x_1), \ldots, f'(x_n)$; there exists a unique homomorphism $\tilde{f}: A' \to B'$ such that $\tilde{f}(x_i) = \varphi_i$; for any polynomial $a \in A$, we have $\pi \circ \tilde{f}(a) = f'(a)$; therefore, for any $a \in A$ we have $\pi \circ \tilde{f}(a) - f'(a) \in \bigcap_k \mathfrak{M}^k(B)$. But B is noetherian, hence separated in the Krull topology; therefore, we have $\bigcap_k \mathfrak{M}^k(B) = \{0\}$, and $\pi \circ \tilde{f} = f'$.

Now, we may consider E as a finite B'-module, and we are reduced to consider the situation $(A', B', \tilde{f}, E)$ instead of the given one; but that case was treated in step 2; this ends the proof of the theorem.

Remarks. 1. The same proof applies to the real case, and, more generally, to analytic algebras over a complete valuated field.

2. In the C^∞ case (over $\mathbf{R}$), it is known that the existence part of theorem 3.1.1. is true. Therefore steps 1 and 2 of the preceeding proof are applicable, but not step 3 (the lifting $\tilde{f}$ cannot be constructed a priori, so one has to *suppose* that such a lifting exists).

3.2. *Germs of analytic spaces.*

This concept will be introduced in terms of categories. As objects, we take triples $(X, \mathcal{O}_X, x)$ where $(X, \mathcal{O}_X)$ is an analytic space, and x a point of X; as morphisms of $(X, \mathcal{O}_X, x)$ into $(Y, \mathcal{O}_Y, y)$ we take the germs at x of morphisms of $(X, \mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$, which map x into y. To simplify the notations, we write (X, x) for $(X, \mathcal{O}_X, x)$.

We shall prove some results on the correspondence between analytic rings and germs of analytic spaces.

Proposition 3.2.1. To any germ (X, x) of an analytic space is associated an analytic ring $\mathcal{O}_{X,x}$. Every analytic ring is obtained in this way. Every morphism $(X, x) \to (Y, y)$ of germs of analytic spaces induces a homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of analytic rings. Conversely every homomorphism $B \to A$ of analytic rings is obtained from a morphism of corresponding germs of analytic spaces; the latter is unique.

Proof. If (X, x) is a germ of analytic spaces, $\mathcal{O}_{X,x}$ is an analytic ring by definition. Now let $A = \mathbf{C}\{x_1, \ldots, x_n\}/\mathrm{I}$ be an analytic ring. We choose generators $f_1, \ldots, f_p$ for I and take an open neighborhood U of 0 such that representatives of $f_1, \ldots, f_p$ which are analytic in U can be found. These generators then define a coherent sheaf $\mathcal{I}$ of ideals on U which defines an analytic subspace X of U with $\mathcal{O}_{X,0} = A$.

If $f : B \to A$ is a homomorphism of analytic rings, we shall construct a morphism $(X, 0) \to (Y, 0)$ of corresponding germs which induces F. We may suppose

$$A = \mathbf{C}\{x_1, \ldots, x_n\}/(f_1, \ldots, f_p), \qquad B = \mathbf{C}\{y_1, \ldots, y_m\}/(g_1, \ldots, g_q);$$

as we have seen in § 1, F can be lifted into a homomorphism $F^1 : \mathbf{C}\{y_1, \ldots, y_m\} \to \mathbf{C}\{x_1, \ldots, x_n\}$; we can choose 1) open sets $U \subset \mathbf{C}^n$, $V \subset \mathbf{C}^m$ with $0 \in U, 0 \in V$ 2) holomorphic functions $\bar{f}_1, \ldots, \bar{f}_p$ in U and $\bar{g}_1, \ldots, \bar{g}_q$ in V such that their germs at 0 are precisely the f_i's and the g_j's, and 3) an holomorphic mapping $\Phi : U \to V$, with $\Phi(0) = 0$ such that Φ^* induces F' at the origin.

Denote now by $\mathcal{I}$ (resp $\mathcal{J}$) the coherent sheaf of ideals generated in U (resp. V) by the $\bar{f}_i's$ | resp. the $\bar{g}'_j s$). We have $\Phi^*(\mathcal{J})_0 \subset \mathcal{I}_0$, hence, since $\mathcal{J}$ is finitely generated by restricting U and V if necessary, we have $\Phi^* (\mathcal{J}) \subset \mathcal{I}$. Finally we take $X = \operatorname{supp} \mathcal{O}_U/\mathcal{I}$, $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I} \mid_X$ and the same for Y ; it is clear that Φ induces the required morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

Finally, if two morphisms $\varphi, \psi : (X, 0) \to (Y, 0)$ induce the same homomorphism $\mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$, we have to prove that φ and ψ are equals. We may assume that Y is given by a local model $(Y, \mathcal{O}_V \mid \mathcal{J} \mid Y)$ for some coherent sheaf $\mathcal{J}$ of ideals on an open set $V \subset \mathbf{C}^m$; by composition with the injection $Y \to V$, we may restrict ourselves to the case where $Y = \mathbf{C}^m$; the morphisms φ and ψ are now given by sections $f, g \in \Gamma(X, \mathcal{O}_X{}^m)$, and the hypothesis means that the germs of f and g at 0 coincide ; hence f and g coincide in a neighborhood of 0 in X, which proves the assertion.

3.3 *Finite morphisms*

Let $f : (X, 0) \to (Y, 0)$ be a morphism of germs of analytic spaces. Then f is called " finite " if the corresponding homomorphism $f^* : \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ makes $\mathcal{O}_{X,0}$ finite over $\mathcal{O}_{Y,0}$. According to the preparation theorem 3.1.3. in order that f be finite, it is necessary and sufficient that $\mathcal{O}_{X,0}/\mathfrak{M}(\mathcal{O}_{Y,0})\, \mathcal{O}_{X,0}$ be finite over $\mathbf{C}$; in geometrical terms, this means that the germ of space $f^{-1}(0)$ is finite over the point 0 (see § 1.3, example 4).

In the global case (complex or real), we give the following definition:

Definition 3.3.1. A morphism of *separated* analytic spaces $f = (f_0, f^1)$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is finite if the following properties hold:

1) f is proper (i.e. f_0 is proper).

2) For any point $x \in X$, the induced morphism of germs $f_x : (X, \mathcal{O}_X, x) \to (Y, \mathcal{O}_Y, f_0(x))$ is finite.

In the *complex* case, we have the following results :

Proposition 3.3.2. f is finite if and only if f is proper and, for any $b \in Y$, the *set* $f_0{}^{-1}(b)$ is finite.

This proposition is more or less equivalent to the " Nullstellensatz "; for the proof see e.g. Houzel [6] or Narasimhan [9]. In the real case, the part " if " of this proposition is not even true when Y is a point : for instance the subspace of $\mathbf{R}^2$ defined by $\mathcal{I}$ = (coherent sheaf of ideals generated by $x_1^2 + x_2^2$) has support 0 ; but $\mathbf{R}\{x_1, x_2\}/(x_1^2 + x_2^2)$ is not finite over $\boldsymbol{R}$.

Proposition 3.3.2. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a finite morphism, then the direct image $f_*(\mathcal{O}_X)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules ; conversely, let $\mathscr{A}$ be a sheaf of $\mathcal{O}_Y$-algebras, which is coherent as sheaf of $\mathcal{O}_Y$-modules. Then there exists an analytic space $(X, \mathcal{O}_X)$ and a finite morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f_* (\mathcal{O}_X)$ is isomorphic with $\mathscr{A}$ as sheaf of $\mathcal{O}_Y$-algebras ; the triple $(X, \mathcal{O}_X, f)$ is unique up to an isomorphism.

We do not prove this proposition here and refer to Houzel [6] or Narasimhan [9] for this proof. We note also that a proof of the direct part can be given along the same lines as theorem 3.1.3, combined with the fact that direct images under finite morphism preserve exact sequences of sheaves of $\mathcal{O}_X$-modules (in other words, that higher direct images are zero). We note also that, for *proper* morphisms (not necessarily finite), a much deeper result has been proved by Grauert [2], [3].

Finally, we remark that, in the real case, proposition 3.3.2. is false (take, for instance, X the submanifold of $\mathbf{R}^2$ defined by $x_2 - x_1^2 = 0$, $Y = \mathbf{R}$ and $f =$ the projection on the x_2-axis ; $f_* (\mathcal{O}_X)$ has support $x_2 \geqslant 0$, which is not an analytic subset of $\mathbf{R}$, hence $f_* (\mathcal{O}_X)$ cannot be coherent!)

Chapter 4.

THE FINITENESS THEOREM

In this chapter, we consider only *complex* analytic spaces, separated and having a countable basis of open sets.

4.1. *Stein spaces*

Let $(X, \mathcal{O}_X)$ be an analytic space, and K a subset of X ; we denote, as usual by $\hat{K}$ the set

$$\left\{ x \in X \mid \forall f \in \Gamma(X, \mathcal{O}_x) : |f(x)| \leqslant \sup_{y \in K} |f(y)| \right\}$$

Definition 4.1.1. a) $(X, \mathcal{O}_X)$ is called holomorphically convex if, for any K compact $\subset X$, $\hat{K}$ is compact ;
b) $(X, \mathcal{O}_X)$ is called a Stein space if it is holomorphically convex, and if, for any $x \in X$, there exist sections $f_1, \ldots, f_p \in \Gamma(X, \mathcal{O}_X)$ with $f_i(x) = 0$, such that x is an isolated point of the counter-image of 0 in the morphism $(X, \mathcal{O}_X) \to \mathbf{C}^p$ defined by $f_1, \ldots, f_p$, (This last property can also by expressed as the fact that the morphism of germs : $(X, \mathcal{O}_X, x) \to (\mathbf{C}^p, 0)$ defined by $f_1, \ldots, f_p$ is finite).

If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (" Theorems A and B " of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space $(X, \mathcal{O}_X)$. Then

1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
2) For $p \geqslant 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let $(X, \mathcal{O}_X)$ be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; if F is an analytic coherent sheaf on X, let $\tilde{F}$ be the trivial extension of F to U ; then $\tilde{F}$ is a coherent sheaf of $\mathcal{O}_U$ modules, and theorems A and B are valid for $\tilde{F}$: therefore, they are true for F.

4.2. *Topology on $\Gamma(X, F)$.*

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; and, with the previous notations, suppose that $\tilde{F}$ admits a *finite presentation* i.e. an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \to 0 .$$

Applying theorem B to the exact sequences

$$0 \to \operatorname{Im}\alpha \to \mathcal{O}_U^p \to \tilde{F} \to 0 \qquad \text{and} \qquad 0 \to \operatorname{Ker}\alpha \to \mathcal{O}_U^q \to \operatorname{Im}\alpha \to 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U,\alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U,\beta)} \Gamma(U, \tilde{F}) \to 0 .$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\operatorname{Im}\Gamma(U, \alpha)$ is closed. For, if f is adherent to $\operatorname{Im}\Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \operatorname{Im}(\alpha_x)$, hence $f \in \Gamma(U, \operatorname{Im}\alpha)$; but, according to theorem B, the mapping $\Gamma(U, \mathcal{O}_U)^q \to \Gamma(U, \operatorname{Im}\alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \operatorname{Im}\Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of $\tilde{F}$ (in fact, it does not even depend on the imbedding $X \to U$, but we shall not need it here). For, suppose we have a second presentation

$$\Gamma(U,\mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U,\mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \to 0\,.$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$-linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta = \beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U,\mathcal{O}_U)^p/\mathrm{Im}\,\Gamma(U,\alpha) \to \Gamma(U,\mathcal{O}_U)^{p'}/\mathrm{Im}\,\Gamma(U,\alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. *General case*

If X is an analytic space and F an analytic coherent sheaf on X, we can find a) a locally finite covering of X by open subspaces X_i, b) for each i, a morphism $X_i \to U_i$, U_i open polycylinder in $\mathbf{C}^{n_i}$, which identifies X_i with a closed subspace of U_i c) for each i, a coherent sheaf $\tilde{F}_i$ on U_i admitting a finite presentation, such that $\tilde{F}_i$ is the extension of $F/_{X_i}$.

On $\Gamma(X_i, F\,|\,_{X_i})$ we have already defined a topology ; further, consider the natural injection

$$\Gamma(X,F) \to \prod_i \Gamma(X_i, F|_{X_i}))$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j$ ($= X_i \times_X X_j$), we have $(f_i)_x = (f_j)_x$; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has tavo coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X, the restriction map $\Gamma(X, F) \to \Gamma(X', F\,|\,_{X'})$ is continuous. If X' is relatively compact in X, then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j, there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i, and applying Ascoli's theorem).

4.3. *Topology on* $H^p(X, F)$

We consider a locally finite covering $\mathcal{U} = \{X_i\}_{i\in I}$ by open subspaces of the preceding type. If we have $i_0, \ldots, i_p \in$ I, we consider the natural morphisms

$$X_{i_0 \ldots i_p} = X_{i_0} \underset{X}{\times} \ldots \underset{X}{\times} X_{i_p} \to X_{i_0} \times \ldots \times X_{i_p} \to U_{i_0} \times \ldots \times U_{i_p}$$

which makes $X_{i_0}, \dots, {}_{i_p}$ isomorphic with a closed subspace of $U_{i_0} \times \dots \times U_{i_p}$ (the hypothesis that X is separated is essential here! See remark at the end of this paragraph), therefore, $X_{i_0}, \dots, {}_{i_p}$ satisfies theorems A and B ; more generally, if a finite number of open subspaces of X is Stein, their intersection is also Stein.

Introduce a total order on I. Given an analytic coherent sheaf on X, we can identify the alternating cochains of degree p of the covering $\mathcal{U}$ with values in F with the space

$$C^p(\mathcal{U},F) = \prod_{i_0 < i_1 < \dots < i_p} \Gamma(X_{i_0 \dots i_p}, F|x_{i_0 \dots i_p}) .$$

This is a Frechet space, and the differential d: $C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$ is clearly continuous. Therefore the kernel $Z^p(\mathcal{U}, F)$ is a closed subspace of $C^p(\mathcal{U}, F)$. We denote $B^p(\mathcal{U},F)$ the image of $C^{p-1}(\mathcal{U},F)$ under d, and we consider on $H^p(\mathcal{U},F) = Z^p(\mathcal{U},F)/B^p(\mathcal{U}, F)$ the quotient topology; according to Leray's theorem, there is a natural isomorphism $H^p(X,F) \simeq H^p(\mathcal{U},F)$.

This gives a topology on $H^p(X, F)$ of a quotient of a Frechet space. In general, this topology is *not separated.*

We prove now that this topology is independent of the covering $\mathcal{U}$; to do that, it is sufficient to consider a refinement $\mathcal{U}' = \{ X'_j \}_{j \in J}$ of $\mathcal{U}$ of the same type, a map $\varphi : J \to I$ such that $X'_j \subset X_{\varphi(j)}$ for any j to consider the map defined by $\varphi : C^*(\mathcal{U},F) = \oplus_p C^p(\mathcal{U},F) \xrightarrow{p} C^*(\mathcal{U}',F)$ and to prove that the induced map $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is an isomorphism.

First, $\bar{\rho}$ is obviously continuous and bijective ; so, according to the closed graph theorem, all that we have to prove is that $\bar{\rho}$ maps the adherence of 0 onto the adherence of zero ; to do that, we consider $\bar{a}' \in H^p(\mathcal{U}, F)$, which is adherent to zero ; this means that $\bar{a}'$ is the class modulo $B^p(\mathcal{U}',F)$ of some $a' \in Z^p(\mathcal{U}' F)$ which is adherent to $B^p(\mathcal{U}', F)$; therefore, we have

$$a' = \lim_{n \to \infty} db'_n , \qquad b'_n \in C^{p-1}(\mathcal{U}', F) .$$

Now, the map

$$Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \overset{(\rho,d)}{\to} Z^p(\mathcal{U}', F)$$

is surjective hence, according to the closed graph theorem, we can find converging sequences $a_n \in Z^p(\mathcal{U}, F)$ and $b''_n \in C^{p-1}(\mathcal{U}', F)$ such that $d\,b'_n = \rho(a_n) + d\,b''_n$; but, $\bar{\rho}$ being an isomorphism, we have $\underline{a_n = d\,\alpha_n}$, $\alpha_n \in C^{n-1}(\mathcal{U}, F)$; if we put $b = \lim_{n \to \infty} b_n$, $a = \lim a_n$, we find that $a \in \overline{B^p(\mathcal{U}, F)}$ and that the class a of a is $H^p(\mathcal{U}, F)$ verifies $\bar{\rho}(\bar{a}) = \bar{a}'$; this proves the result.

Remark. If X is not separated, an intersection of two open Stein subspaces of X need not be Stein; take f.i. for X two copies of C^2, identified everywhere except at O; there is an obvious covering of X by two open subspaces, identicals with C^2; but their intersection is $C^2 - \{o\}$, and therefore is not Stein!

4.4. *The finiteness theorem*

Theorem 4.4.1. (Cartan — Serre). Let X be a compact analytic space, and F be a coherent analytic sheaf on X. Then, for every $p \geqslant 0$ $H^p(X, F)$ is separated and finite dimensional.

We shall give two proofs of this theorem ; both are interesting for further applications.

1st proof. Let $\{ X_i \}$ and $\{X_i'\}$ be two finite coverings of X of the type considered in the previous articles, such that, for every i, X'_i is relatively compact in X_i. Then, if we denote by $\mathcal{U}$ (resp. $\mathcal{U}'$) the covering $\{X_i\}$ (resp. $\{ X_i' \}$), the natural restriction map $C^p(\mathcal{U}, F) \to C^p(\mathcal{U}', F)$ is compact.

Consider now the map

$$(\rho, d) : Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \to Z^p(\mathcal{U}', F)$$

this map is surjective, and we have $(, d\rho) = (\rho, 0) + (0, d)$, $(\rho, 0)$ being compact ; then the following lemma proves that Im $(0, d)$ is closed and finite codimensional, q.e.d.

Lemma 4.4.2. Let E and F two Frechet spaces, u_1 and u_2 two linear continuous maps $E \to F$ such that $u_1 + u_2$ is surjective, and u_1 compact. Then Im (u_2) is closed and finite codimensional. For the proof, see e.g. [5].

2nd proof. Consider $\mathcal{U}$ and $\mathcal{U}'$ as above, and consider the map
(ρ, d) $C^{p-1}(\mathcal{U}, F)/Z^{p-1}(\mathcal{U}, F) \to [C^{p-1}(\mathcal{U}', F)/Z^{p-1}(\mathcal{U}', F)] \oplus Z^p(\mathcal{U}, F)$
(ρ, d) is clearly injective. I claim that its image is closed: In fact, since $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is injective, this image consists of the pairs $(\bar{a}', b)$, $a' \in C^{p-1}(\mathcal{U}', F)$, $b \in Z^p(\mathcal{U}, F)$ such that $da' = \rho\, b$, which proves the assertion.

Now we have $(\rho, d) = (\rho, 0) + (0, d)$ and $(\rho, 0)$ is compact. By a well-known lemma, it results that Im $(0, d)$ is closed, which means that $H^p(\mathcal{U}, F)$ is separated.

Finally, since $\bar{\rho}$ is compact, and is an isomorphism, it follows that the identity map of $H^p(\mathcal{U}, F)$ into itself is compact ; therefore this space is finite dimensional ; this proves the theorem.

Appendix

Local noetherian rings

All rings here as well as in the preceding lectures are supposed to be commutative and have units. A ring A is called *local* if it contains exactly one maximal ideal ; this will be denoted by $\mathfrak{m}(A)$ or simply $\mathfrak{m}$. A module E over a ring A is called *finite* if it is finitely generated over A. A module E over A is called *noetherian* if E is unitary (i.e. $1\,x = x$ for all $x \in E$) and every submodule of E is finite over A. In particular A itself is noetherian if and only if all its ideals are finitely generated.

We state without proof the following result.

Theorem (Lemma of Artin-Rees). Let A be a noetherian ring and I an ideal of A. Let E be a finite A-module and F_1, F_2 submodules of E. Then there is an integer n such that

$$I\left((I^n F_1) \cap F_2\right) = (I^{n+1} F_1) \cap F_2\,.$$

The proof may be found in Nagata [8, Theorem 3.7] or, for $F_1 = E$ which is the only case we shall need, in Bourbaki [1, Ch. III, 3, no 1].

Lemma (called Nakayama's lemma by Bourbaki). Let A be a local ring with maximal ideal $\mathfrak{m}$ and E a finite A-module.

(i) If $E = \mathfrak{m}\,E$, then $E = 0$.

(ii) If F is a submodule of E such that $E = F + \mathfrak{m}\,E$, then $F = E$.

(iii) Let $k = A/\mathfrak{m}$, a field. Then $k \otimes_A E = 0$ implies $E = 0$.

Proof. (i) Let $x_1, \ldots, x_n$ be generators for E, $n \geqslant 1$. We can then write $x_n = \sum^n_1 a_j x_j$ for some $a_j \in \mathfrak{m}$, hence $(1 - a_n)\,x_n = \sum_1^{n-1} a_j x_j$. Since $1 - a_n$ is invertible this means that $x_1, \ldots, x_{n-1}$ generate E. The minimal number of generators must therefore be zero, i.e. $E = 0$.

(ii) We only need to apply (i) to E/F.

(iii) We have $k \otimes_A E = E/\mathfrak{m}E$ which reduces (iii) to (i).

If E is a module over a local ring A the sets $\mathfrak{m}^k E$, $k \geqslant 0$, form a basis of the neighborhoods of $0 \in E$ for a topology in E. This topology, making E into a topological group, is called the Krull topology of E.

Combining the two previous results we can prove the following

Theorem (Krull). Let A be a local noetherian ring, E a finite module over A. Then:

(i) The Krull topology of E is separated.

(ii) Every submodule F of E is closed in E.

(iii) The topology induced by E in a submodule F is the Krull topology of F.

Proof. (i) Let $F = \bigcap_{k \geq 0} \mathfrak{m}^k E = \overline{\{0\}}$. Then

$$\mathfrak{m}F = \mathfrak{m}\big((\mathfrak{m}^n E) \cap F\big) = (\mathfrak{m}^{n+1} E) \cap F = F$$

by the Artin-Rees lemma. Hence Nakayama's lemma implies that $F = \{ 0 \}$.

(ii) Let $f : E \to E/F$ be the natural map. Then

$$f(\bar{F}) \subset f(F + \mathfrak{m}^k E) = f(\mathfrak{m}^k E) = \mathfrak{m}^k (E/F)\,.$$

Hence $f(\bar{F}) \subset \cap\, \mathfrak{m}^k (E/F) = \{ 0 \}$, using (i).). But $f(\bar{F}) \subset \{ 0 \}$ is equivalent to $\bar{F} \subset F$.

(iii) It is clear that $\mathfrak{m}^k F \subset (\mathfrak{m}^k E) \cap F$. Hence the Krull topology of F is finer than that induced by E ; in other words the inclusion $F \to E$ is continuous. Conversely the Artin-Rees lemma shows that

$$(\mathfrak{m}^{n+k} E) \cap F = \mathfrak{m}^k \big((\mathfrak{m}^n E) \cap F\big) \subset \mathfrak{m}^k F$$

which proves that the induced topology is finer than the Krull topology of F.

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