4. Other conditions

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4. Other conditions

By theorem 1 both U and U_p vary dominatedly whenever $\underline{r}>0$ and $\bar{r}<\infty$. The next theorem gives even simpler criteria for dominated variation that remain applicable in the limiting situations $\underline{r}=0$ and $\bar{r}=\infty$.

THEOREM 3. *If there exists a number $\xi>1$ such that*

$$\liminf \frac{U(\xi t)}{U(t)} > 1 \tag{4.1}$$

then U_p varies dominatedly. Similarly the relation

$$\liminf \frac{U_p(t/\xi)}{U_p(t)} > 1 \tag{4.2}$$

implies the dominated variation of U.

PROOF. Clearly

$$\frac{t^p\, U_p(t)}{U(t)} \geqslant \frac{t^p\left[U_p(t)-U_p(t\xi)\right]}{U(t)} \geqslant \xi^{-p}\,\frac{U(t\xi)-U(t)}{U(t)}\,. \tag{4.3}$$

When the right side is bounded away from 0 this implies $\underline{r}>0$, and so U_p varies dominatedly by theorem 1.

We can go a step further. If, besides (4.1), it is known that U varies dominatedly with exponent $\gamma<p$, then $t^{-p}\,U(t)/U_p(t)$ is bounded away from 0, and hence the second inequality in (4.3) implies that

$$\liminf \frac{U_p(t)-U_p(t\xi)}{U_p(t)} > 0\,. \tag{4.4}$$

This is equivalent to (4.2).

Similarly

$$\frac{U_p(t)-U_p(t\xi)}{U_p(t\xi)} \leqslant t^{-p}\,\frac{U(t\xi)-U(t)}{U_p(t\xi)} = \tag{4.5}$$

$$= \frac{\xi^p}{R_U(t\xi)}\,\frac{U(t\xi)-U(t)}{U(t\xi)}\,.$$

The second fraction on the right does not exceed 1, and so (4.2) ensures that $R_U(t\xi)$ remains bounded, and hence that U is of dominated variation.

Again, if it is known that R_U is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If U is of dominated variation with exponent $\gamma<p$ then (4.1) implies (4.2). Similarly, if U_p is of dominated variation with exponent $-q$ where $q<p$, then (4.2) entails (4.1).* (In each case both functions are of dominated variation.)

5. RATIO LIMIT THEOREMS

Let U and V be non-decreasing unbounded functions, and suppose that L is slowly varying (= regularly varying with exponent 0).

DEFINITION. *We shall say that U and V are L-equivalent and write*

$$V \leftrightarrow UL \tag{5.1}$$

if the ratio UL/V tends to 1 *at all points of continuity.*

More precisely, it is required that for each $\varepsilon>0$ and fixed $\lambda>1$

$$(1-\varepsilon)\, L(t)\, U(t/\lambda) \leqslant V(t) \leqslant (1+\varepsilon)\, L(t)\, U(t\lambda) \tag{5.2}$$

for all t sufficiently large.

THEOREM 4. *Let U be of dominated variation. In order that there exist a slowly varying function* L *such that (5.1) holds it is necessary and sufficient that*

$$R_U(t) - R_V(t) \to 0 \qquad \textit{boundedly.} \tag{5.3}$$

Needless to say, R_V and $\mathscr{I}_V$ are defined by analogy with R_U in (1.5) and $\mathscr{I}_U$ in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that U satisfies the basic inequality (2.2). Obviously the slow variation of L implies that for t sufficiently large and all $x>1$

$$\frac{V(tx)}{V(t)} < C' x^{\gamma'} \tag{5.4}$$

for any pair of constants $C'>C$ and $\gamma'>\gamma$. Thus V is of dominated variation, and since $p>\gamma$ the function V_p exists.