Chapter I Operators with constant coefficients

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **17 (1971)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

kevič; Mizohata-Ohya [1] and Flaschka-Strang [1] (hyperbolic operators with characteristics of constant multiplicity). The methods discussed in Chapter III can obviously be used to push much further in this direction. For the constant coefficient case a model result is given by Theorem 1.5.1.

Chapter I

Operators with constant coefficients

1.1. *Fundamental solutions*

A differential operator with constant coefficients in $\mathbf{R}^n$ can be written in the form $P(D)$ where P is a polynomial in n variables with complex coefficients and $D = (-i\partial/\partial x_1, \dots, -i\partial/\partial x_n)$. Explicitly

$$P(D) = \sum a_\alpha D^\alpha$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index and the sum is finite.

It is easy to show that the equation

$$P(D)\,u = f \tag{1.1.1}$$

can always be solved locally. To do so we assume first that $f \in C_0^\infty$. If u is a solution of (1.1.1) with a well defined Fourier transform $\hat{u}$, we must have $P(\xi)\,\hat{u}(\xi) = \hat{f}(\xi)$, and so by Fourier's inversion formula

$$u(x) = (2\pi)^{-n} \int e^{i<x,\xi>} \hat{f}(\xi)/P(\xi)\,d\xi\,. \tag{1.1.2}$$

However, P may have zeros in or near $\mathbf{R}^n$ and this makes it necessary to deform the integration contour in order to obtain a well defined solution from (1.1.2).

First note that if $\Phi \in C_0^\infty(\mathbf{C}^n)$ and

$$\Phi(e^{i\theta}\zeta) = \Phi(\zeta)\,,\quad \theta \in \mathbf{R}\,,\quad \int \Phi(\zeta)\,d\lambda(\zeta) = 1\,, \tag{1.1.3}$$

where $d\lambda$ is the Lebesgue measure in $\mathbf{C}^n$, then

$$\int F(\zeta)\,\Phi(\zeta)\,d\lambda(\zeta) = F(0) \tag{1.1.4}$$

for any entire analytic function F. In fact, by Cauchy's integral formula

$$\int F(\zeta e^{i\theta})\,d\theta = 2\pi F(0)\,,$$

and if we multiply by $\Phi(\zeta)$ and integrate, (1.1.3) gives (1.1.4).

Let $Pol\,(m)$ be the complex vector space of polynomials of degree $\leqq m$ and let $Pol^0\,(m)$ be the vector space with the origin removed. If Ω is a neighborhood of 0 in $\mathbf{C}^n$ one can find a C^∞ map $\Phi : Pol^0\,(m) \to C_0^\infty\,(\Omega)$ which is homogeneous of degree zero, such that the range consists of functions satisfying (1.1.3) and for some constant C

$$(1.1.5)\quad \sum | Q^{(\alpha)}(0) | \leqq C\,| Q(\zeta) |\,,\ \ Q \in Pol^0(m)\,,\ \ \zeta \in \operatorname{supp} \Phi\,(Q)\,.$$

Here $Q^{(\alpha)}\,(\xi) = (iD)^\alpha\, Q$; the left hand side is of course a norm in $Pol\,(m)$. For a fixed Q the existence of such a Φ is quite obvious for we can find $\theta \in \mathbf{R}^n$ such that $Q\,(z\theta) \neq 0$ when $|z| = 1$, and (1.1.5) is then fulfilled if the support of Φ is near this circle. The same Φ can be used for all Q near by, and since functions satisfying (1.1.3) form a convex set the construction of Φ can be finished by means of a partition of unity in the set of all Q with $\sum | Q^{(\alpha)}\,(0) | = 1$.

We now replace (1.1.2) by the expression

$$(1.1.6)\quad (Ef)\,(x) = (2\pi)^{-n} \int d\xi \int e^{i<x,\xi+\zeta>}\, \hat{f}(\xi+\zeta)/P\,(\xi+\zeta)\;\Phi\,(P_\xi,\zeta)\, d\lambda\,(\zeta)$$

where P_ξ is the polynomial $\zeta \to P\,(\xi+\zeta)$. Since some derivative of P is a constant, the function

$$(1.1.7)\qquad \tilde{P}\,(\xi) = \sum | P^{(\alpha)}(\xi) |$$

has a positive lower bound. Hence it follows from (1.1.5) that P is bounded away from 0 in the support of the integrand, so (1.1.6) is well defined for $f \in C_0^\infty$. Differentiation under the integral sign gives $P\,(D)\,Ef = f$ in view of (1.1.4) and Fourier's inversion formula. Hence we have solved (1.1.1) when $f \in C_0^\infty\,(\mathbf{R}^n)$. The map $f \to Ef$ commutes with translations so there is a distribution which we also denote by E for which $Ef = E * f$. Since $\big(P\,(D)\,E\big) * f = f$ for all $f \in C_0^\infty$ we have $P\,(D)\,E = \delta$, the Dirac measure at 0. To solve (1.1.1) for arbitrary $f \in \mathscr{E}'\,(\mathbf{R}^n)$, the space of distributions with compact support, it is therefore sufficient to choose $u = E * f$. One calls E a *fundamental solution.*

The preceding construction gives a fundamental solution with optimal local regularity properties (cf. Hörmander [1, section 3.1] where references to earlier literature are also given). The construction is clearly applicable without change if P depends on parameters (cf. Trèves [8], [9]). Summing up:

THEOREM 1.1.1. *There exists a continuous map $E\colon Pol^0\,(m) \to \mathscr{D}'\,(\mathbf{R}^n)$ such that $P\,(D)\,E\,(P) = \delta$ for every $P \in Pol^0\,(m)$.*

1.2. *Global existence theorems*

Let X be an open set in $\mathbf{R}^n$ and let $C^\infty(X)$, $\mathscr{D}'(X)$, $\mathscr{D}'^F(X)$ be the set of all infinitely differentiable functions, distributions and distributions of finite order in X. We shall consider the equation

$$P(D)u = f \tag{1.2.1}$$

with u and f in one of these spaces. Since f may then be very large at the boundary, conditions have to be imposed on X and on P.

THEOREM 1.2.1. *The following four conditions are equivalent:*

(i) *For every* $f \in C^\infty(X)$ *there is a solution* $u \in C^\infty(X)$ *of* (1.2.1).

(ii) *For every* $f \in \mathscr{D}'^F(X)$ *there is a solution* $u \in \mathscr{D}'^F(X)$ *of* (1.2.1).

(iii) *For every* $f \in C^\infty(X)$ *there is a solution* $u \in \mathscr{D}'(X)$ *of* (1.2.1).

(iv) *For every compact set* $K \subset X$ *there is a compact set* $K' \subset X$ *such that*

$$v \in \mathscr{E}'(X)\,, \ \operatorname{supp} P(-D)v \subset K \Rightarrow \operatorname{supp} v \subset K'\,. \tag{1.2.2}$$

The theorem is essentially due to Malgrange [1] (see also Hörmander [1, section 3.5]). Since the proof just consists of abstract functional analysis the equivalence of (i) and (iv) remains valid with minor changes of (iv) if P is a differential operator with variable coefficients for which a semi-global existence theory is established. The operator $P(-D)$ in (1.2.2) should of course be replaced by the formal adjoint ${}^tP$ then. When $f \in \mathscr{D}'(X)$ we have similar results:

THEOREM 1.2.2. *Suppose that* $P(D)$ *defines a surjective map* $\mathscr{D}'(X) \to \to \mathscr{D}'(X)/C^\infty(X)$. *For every compact set* $K \subset X$ *there is then a compact set* $K' \subset X$ *such that*

$$v \in \mathscr{E}'(X)\,, \ \text{sing supp } P(-D)v \subset K \Rightarrow \text{sing supp } v \subset K'\,. \tag{1.2.3}$$

Here sing supp v denotes the smallest closed set such that $v \in C^\infty$ in the complement. Although Theorem 1.2.2 is not formally identical to Theorem 3.6.3 in Hörmander [1], the proof of that theorem is actually a proof of Theorem 1.2.2 above. A similar result is sometimes but not always valid for operators with variable coefficients.

Example 1.2.3. For the differential operator $P = \sin \pi x\, d/dx$ on $\mathbf{R}$ we have $P\mathscr{D}'(\mathbf{R}) = \mathscr{D}'(\mathbf{R})$. In fact, to solve the equation $Pu = f$ we have only to solve first a simple division problem and then an ordinary differential equation. However, ${}^tPv = 0$ for all measures v supported by the integers so the analogue of (1.2.3) would be false.

On the other hand, the converse of Theorem 1.2.2 is very general:

THEOREM 1.2.4. *Let X be a C^∞ manifold and P a continuous linear map $\mathscr{D}'(X) \to \mathscr{D}'(X)$ whose restriction to $C^\infty(X)$ is a (continuous) map into $C^\infty(X)$. Denote by ${}^tP$ the adjoint with respect to some positive density in X, which is then a continuous operator in $C_0^\infty(X)$ and in $\mathscr{E}'(X)$. Assume that to every compact set K in X there is another compact set K' in X, which can be taken empty if K is empty, such that*

$$(1.2.3)' \quad v \in \mathscr{E}'(X)\,, \ \text{sing supp}\ {}^tPv \subset K \Rightarrow \text{sing supp}\ v \subset K'\,.$$

Then P defines a surjective map $\mathscr{D}'(X) \to \mathscr{D}'(X)/C^\infty(X)$.

From Theorems 1.2.1, 1.2.2 and 1.2.4 we obtain

COROLLARY 1.2.5. *If X is an open set in $\mathbf{R}^n$ we have $P(D)\,\mathscr{D}'(X) = \mathscr{D}'(X)$ if and only if to every compact set $K \subset X$ there is another compact set $K' \subset X$ such that* (1.2.2) *and* (1.2.3) *are valid.*

Corollary 1.2.5 was proved in section 3.6 of Hörmander [1]. A proof of Theorem 1.2.4 is easily extracted from the proof of Theorem 3.6.4 there, but we give it in full here as a typical case of the arguments relating theorems on existence of solutions to theorems on regularity of solutions.

Proof of Theorem 1.2.4. It is sufficient to prove that for every $f \in \mathscr{D}'(X)$ there is a continuous semi-norm q on $C_0^\infty(X)$ and a sequence $\psi_j \in C_0^\infty(X)$ with locally finite supports such that

$$(1.2.4) \qquad |f(\varphi)| \leqq q({}^tP\varphi) + \sum |<\varphi, \psi_j>|\,, \quad \varphi \in C_0^\infty(X)\,.$$

In fact, if we apply the Hahn-Banach theorem to extend the map

$$({}^tP\varphi,\ <\varphi,\psi_1>,\ <\varphi,\psi_2>, \ldots) \to f(\varphi)$$

to a linear form on $C_0^\infty(X) \oplus l^1$, we obtain an element $u \in \mathscr{D}'(X)$ and a bounded sequence a_j such that

$$f(\varphi) = u({}^tP\varphi) + \sum a_j <\varphi, \psi_j>\,, \quad \varphi \in C_0^\infty(X)\,,$$

which means that $f = Pu + \sum a_j \psi_j$. To prove (1.2.4) we first replace $|f(\varphi)|$ by an arbitrary continuous semi-norm $F(\varphi)$ in $C_0^\infty(X)$ which is stronger than the maximum norm for example. We want to prove that

$$(1.2.4)' \qquad F(\varphi) \leqq C\big(q({}^tP\varphi) + \textstyle\sum |<\varphi,\psi_j>|\big)\ ,\ \varphi \in C_0^\infty(X)\,.$$

Choose an increasing sequence K_j of compact sets in X with union X and $K_0 = \varnothing$ and choose for every j a corresponding K_j' according to the hypothesis so that $K_0' = \varnothing$ and K_j' is in the interior of K_{j+1}'. (Note that we require manifolds to be countable at infinity.)

LEMMA 1.2.6. *Assume that* (1.2.4)′ *is valid when* $\varphi \in C_0^\infty(K_j')$. *If* $\varepsilon > 0$ *one can find another semi-norm* q' *on* C_0^∞ *such that* $q'(\psi) = q(\psi)$ *when* $\psi \in C_0^\infty(K_{j-1})$ *and* (1.2.4)′ *is valid when* $\varphi \in C_0^\infty(K_{j+1}')$ *if* C *is replaced by* $(1+\varepsilon)\,C$, q *is replaced by* q' *and the functions* ψ_j *are supplemented by a finite number of functions in* $C_0^\infty(\complement K_{j-1}')$.

If we note that the hypothesis of the lemma is trivially fulfilled when $j = 0$ and if we iterate this conclusion with a sequence ε_j with $\Pi(1+\varepsilon_j) < \infty$, we conclude from the lemma that (1.2.4)′ is valid for suitable C, q and ψ_j.

Proof of Lemma 1.2.6. Let Φ be the completion of $C_0^\infty(K_{j+1}')$ in the weakest topology in which $F(\varphi)$ is continuous and the map from φ to the restriction of ${}^tP\varphi$ to $\complement K_{j-1}$ is continuous with values in $C^\infty(\complement K_{j-1})$. Then Φ is contained in the space of continuous functions with support in K_{j+1}', and for every $\varphi \in \Phi$ we have ${}^tP\varphi \in C^\infty(\complement K_{j-1})$, hence $\varphi \in C^\infty(\complement K_{j-1}')$. It follows that restricting functions in Φ to $\complement K_{j-1}'$ gives a continuous map from Φ to $C^\infty(\complement K_{j-1}')$ so bounded sequences in Φ are also bounded in the latter space.

Let $\chi_1, \chi_2, \ldots$ be a dense sequence in $C_0^\infty(\complement K_{j-1}')$, and let $q_1, q_2, \ldots$ be semi-norms defining the topology in $C^\infty(\complement K_{j-1})$. For convenience we choose these so that $2q_j \leqq q_{j+1}$ for every j. Then we claim that for some integer N and all $\varphi \in C_0^\infty(K_{j+1}')$

(1.2.5)

$$F(\varphi) \leqq C(1+\varepsilon)\big(q({}^tP\varphi) + \textstyle\sum |<\varphi,\psi_j>| + q_N({}^tP\varphi) + N \displaystyle\sum_{k<N} |<\varphi,\chi_k>|\big)\,.$$

This would prove the lemma. Now if (1.2.5) is not valid for any N we can choose a sequence $\varphi_N \in C_0^\infty(K_{j+1}')$ such that

$$F(\varphi_N) = C(1+\varepsilon)\,,\ q({}^tP\varphi_N) + \textstyle\sum |<\varphi_N, \psi_j>| \leqq 1$$

and ${}^tP\varphi_N \to 0$ in $C^\infty(\complement K_{j-1})$, $<\varphi_N, \chi_k> \to 0$ for every k as $N \to \infty$. But then φ_N is relatively compact in $C^\infty(\complement K_{j-1}')$ and every limit is orthogonal to all χ_k and therefore equal to 0. Thus $\varphi_N \to 0$ in $C^\infty(\complement K'_{j-1})$. Choose now a function $\psi \in C_0^\infty(K'_j)$ which is 1 in a neighborhood of K'_{j-1}. Then it follows that $(1-\psi)\varphi_N \to 0$ in C_0^∞. If $\varphi'_N = \psi\varphi_N$ we obtain for large N

$$F(\varphi'_N) > C(1+2\varepsilon/3)\,,\ q({}^tP\varphi'_N) + \sum_j | < \varphi'_N, \psi_j > | < 1 + \varepsilon/3\,.$$

Since $\varphi'_N \in C_0^\infty(K'_j)$, this contradicts the hypothesis that (1.2.4) is valid for such functions. The proof is complete.

It is a simple exercise in Fredholm theory to show that the hypotheses of Theorem 1.2.4 imply that for every compact set $K \subset X$ the space $N(K)$ of all $v \in \mathscr{E}'(K)$ with ${}^tPv = 0$ is finite dimensional, and that the equation $Pu = f \in \mathscr{D}'(X)$ can be fulfilled on a neighborhood of K with $u \in \mathscr{D}'(X)$ if (and only if) f is orthogonal to $N(K)$. In fact, for this we only need that sing supp ${}^tPv = \varnothing$ implies sing supp $v = \varnothing$ when $v \in \mathscr{E}'(X)$. Thus results on the regularity of solutions of differential equations imply theorems on the existence of solutions, and for this reason we shall mainly pay attention to the regularity of solutions in these lectures.

Returning to differential operators with constant coefficients we introduce a slight modification of the terminology in Hörmander [1].

Definition 1.2.7. The open set X in $\mathbf{R}^n$ is called P-convex with respect to supports (resp. singular supports) if for every compact set $K \subset X$ there is another compact set $K' \subset X$ such that (1.2.2) (resp. (1.2.3)) is valid.

The use of the term " convex " will be justified by the discussion of the geometric meaning in sections 1.3 and 1.4. Here we just note that convex sets are P-convex both with respect to supports and singular supports. An elementary argument using the translation invariance of $P(-D)$ also gives (see Theorem 3.5.2 in Hörmander [1]):

THEOREM 1.2.8. *Let $x \to |x|$ denote any norm in $\mathbf{R}^n$ and set for closed sets F in X*

$$d(F, \complement X) = \inf_{x \in F, y \notin X} |x - y|\,.$$

Then X is P-convex with respect to supports if and only if

(1.2.6) $\qquad d(\operatorname{supp} P(-D)v, \complement X) = d(\operatorname{supp} v, \complement X), v \in \mathscr{E}'(X)\,,$

and with respect to singular supports if and only if

(1.2.7) $\qquad d(\text{sing supp } P(-D)v, \complement X) = d(\text{sing supp } v, \complement X)\,, v \in \mathscr{E}'(X)\,.$

The analogy of the notions of P-convexity in Definition 1.2.7 to holomorphic convexity in the theory of functions of several complex variables is obvious. The purpose of the next two sections is to discuss some analogues of pseudo-convexity.

1.3. *Geometric conditions for P-convexity with respect to supports*

Throughout this section we denote by X an open set in $\mathbf{R}^n$ and by $P(D)$ a partial differential operator with constant coefficients. The following two simple theorems describe the conditions for P-convexity of X which involve only P or only X.

THEOREM 1.3.1. *X is P-convex with respect to supports for every P if and only if every component of X is convex in the usual sense.*

THEOREM 1.3.2. *Every X is P-convex with respect to supports if and only if P is elliptic.*

Ellipticity means, if P is of degree m and

$$P(\xi) = P_m(\xi) + P_{m-1}(\xi) + \dots$$

is the decomposition of P in a sum of homogeneous terms P_j of degree j, that

$$P_m(\xi) \neq 0 \quad \textit{if} \quad 0 \neq \xi \in \mathbf{R}^n . \tag{1.3.1}$$

P_m is called the principal part of P. Solutions of the equation $P_m(\xi) = 0$ with $\xi \neq 0$ (and $\xi \in \mathbf{R}^n$) are called (real) *characteristics*. A hypersurface is said to be characteristic when the normal is characteristic. A characteristic point ξ with $dP_m(\xi) \neq 0$ is said to be simply characteristic, and the projection in $\mathbf{R}^n$ of a complex line in $\mathbf{C}^n$ with direction $(\partial P_m/\partial \xi_1, \dots, \partial P_m/\partial \xi_n)$ will then be called a *bicharacteristic* corresponding to ξ. It may be of dimension 1 or 2.

Now observe that X is not P-convex with respect to supports if for some open set $Y \subset\subset X$ (i.e. Y is relatively compact in X) there is a distribution $u \in \mathscr{D}'(Y)$ with

$$d(\operatorname{supp} u, \complement X) < \min\left(d(\partial Y \cap \overline{\operatorname{supp} u}, \complement X), d(\operatorname{supp} P(-D)\, u, \complement X)\right). \tag{1.3.2}$$

In fact, (1.2.6) is not valid if $v = \varphi u$ and $\varphi \in C_0^\infty(Y)$ is equal to 1 in a sufficiently large compact subset of Y. If $P(-D)u = 0$ in Y and $u = 0$ at a part of the boundary this leads to necessary conditions for P-convexity. In particular one can use the fact that there is a solution of $P(-D)u = 0$ with support equal to any half space with characteristic boundary (Hörmander [1, Theorem 5.2.2]). In stating the result we shall say that a function f in X satisfies the minimum principle in a closed set F if for every compact set $K \subset F \cap X$ we have

$$\min_{x \in K} f(x) = \min_{x \in \partial_F K} f(x)$$

where $\partial_F K$ is the boundary of K as a subset of F. We write $d_X(x) = d(\{x\}, \complement X)$.

THEOREM 1.3.3. *If X is P-convex with respect to supports, then $d_X(x)$ satisfies the minimum principle in any characteristic hyperplane. When $n = 2$ this means that every component of X is convex in the direction of any characteristic line, and this condition is also sufficient for X to be P-convex with respect to supports.*

For the proof we refer to section 3.7 in Hörmander [1], where it is also shown that Theorem 1.3.3 implies the necessity in Theorems 1.3.1 and 1.3.2. When $n > 2$ the necessary condition in Theorem 1.3.3 is far from sufficient, however, for there are many characteristic surfaces which are not planes and a classical theorem of Goursat allows one to construct local solutions vanishing on one side of any simply characteristic surface. Thus Malgrange [2] proved (see also Theorem 3.7.3 in Hörmander [1]):

THEOREM 1.3.4. *Let $P(D)$ be a differential operator such that the principal part $P_m(D)$ has real coefficients and let X be P-convex with respect to supports. At every simply characteristic C^2 boundary point the normal curvature of ∂X in the direction of the corresponding bicharacteristic must then be non-negative.*

Actually the proof of Malgrange gives somewhat more, namely that for no boundary point x_0 with simply characteristic normal N_0 does there exist a cylinder with C^2 boundary and the corresponding bicharacteristic as generator containing x_0 and contained in $X \cup \{x_0\}$ near x_0. This improvement is given in a different form in Trèves [1].

In the proof of Theorem 1.3.4 a simply characteristic surface is constructed by means of the Hamilton-Jacobi integration theory. Using this theory for the system of equations Re P_m (grad φ) $= 0$, Im P_m (grad φ) $= 0$ (see e.g. Carathéodory [1, Chapter IV]) one obtains

THEOREM 1.3.5. *Let X be P-convex with respect to supports and have a C^2 boundary. At every boundary point where the normal is simply characteristic and the corresponding bicharacteristic is two dimensional the normal curvature of ∂X in some direction in the bicharacteristic must then be non-negative.*

So far we have only given necessary conditions for P-convexity. To give sufficient conditions means to prove uniqueness theorems. For example, the sufficiency in Theorem 1.3.2 follows from the fact that solutions of homogeneous elliptic equations are real analytic and therefore have a property of unique continuation. In general we have available the uniqueness theorem of Holmgren (see Hörmander [1, section 5.3]) and variations of it for continuation across characteristic surfaces. (See Trèves [1], Zachmanoglou [1], Bony [1], Hörmander [11, 12].) From the results of Hörmander [11] we obtain, for example, the following theorem which should be compared with Theorem 1.3.4; it is clear that an analogous result can be proved corresponding to Theorem 1.3.5.

THEOREM 1.3.6. *Let $P(D)$ be a differential operator such that the principal part $P_m(D)$ has real coefficients, and let X be an open set in $\mathbf{R}^n$ with a C^1 boundary. Then X is P-convex with respect to supports if every characteristic boundary point x_0 is simple and for every closed interval I on the corresponding bicharacteristic with $x_0 \in I \subset \overline{X}$ at least one end point belongs to ∂X.*

The proof of Theorem 3.7.3 in Hörmander [1] gives the following partial converse of Theorem 1.3.5 involving weaker conditions on P and stronger conditions on X:

THEOREM 1.3.7. *Let X have a C^2 boundary for which all characteristic points with respect to P are simple. Assume that at every characteristic boundary point the normal curvature of ∂X in some direction in the corresponding bicharacteristic is positive. Then it follows that X is P-convex with respect to supports.*

For later reference we give a simple modification of Theorem 1.3.2:

THEOREM 1.3.8. *Let $P(D)$ be a differential operator in $\mathbf{R}^n$ which acts along a linear subspace V and is elliptic as an operator in V. Then an open set X in $\mathbf{R}^n$ is P-convex with respect to supports if and only if $d_X(x)$ satisfies the minimum principle in any affine space parallel to V.*

This ends our quite fragmentary list of results. It is clear that P-convexity with respect to supports is insufficiently understood as yet. Further study should lead to improved uniqueness theorems.

1.4. *Geometric conditions for P-convexity with respect to singular supports*

As in section 1.3 we denote throughout by X an open set in $\mathbf{R}^n$ and by $P(D)$ a partial differential operator with constant coefficients. Again we start by describing the convexity conditions which only involve P or X.

THEOREM 1.4.1. *X is P-convex with respect to singular supports for every P if and only if every component of X is convex in the usual sense.*

THEOREM 1.4.2. *Every X is P-convex with respect to singular supports if and only if P is hypoelliptic.*

Hypoellipticity means that for every distribution u

$$\text{sing supp}\, u = \text{sing supp}\, Pu \tag{1.4.1}$$

or equivalently that (Hörmander [1, section 4.1])

$$P^{(\alpha)}(\xi)/P(\xi) \to 0 \text{ when } \xi \to \infty \text{ in } \mathbf{R}^n \text{ if } |\alpha| \neq 0 . \tag{1.4.2}$$

The sufficiency is well known (see section 3.7 in Hörmander [1]) in Theorem 1.4.1 and is trivial in Theorem 1.4.2. The necessity will follow from more precise results below.

Necessary conditions for P-convexity with respect to singular supports can be obtained by noting that X is not P-convex in this sense if (1.3.2) is valid for some u and $Y \subset\subset X$ with supports replaced by singular supports. To use this remark we need to know solutions of the equation $P(-D)\,u = 0$ with small singular support. Starting from earlier constructions by Zerner [1] and Hörmander [1, section 8.8] rather general results of this type were proved in Hörmander [7]. A heuristic motivation for these is obtained by noting that for functions represented as Fourier integrals it is the high frequency components that may give rise to singularities. It is therefore natural to consider solutions of the equation $P(D)u = 0$ of the form $u(x) = e^{i<x,\xi>} v(x)$ where ξ is large and a major part of v is composed of exponentials with much smaller frequencies. We have

$$P(D)(e^{i<x,\xi>}v) = e^{i<x,\xi>} P_\xi(D) v$$

where $P_\xi(D) = P(D+\xi)$. With $\tilde{P}(\xi)$ defined by (1.1.7) the normalized polynomials $P_\xi/\tilde{P}(\xi)$ belong to the unit sphere in $Pol(m)$, if m is the degree of P. Denote the set of limit points when $\xi \to \infty$ by $L(P)$. It is then natural to expect connections between singular supports of solutions of the equation $P(D)u = 0$ and supports of solutions of $Q(D)u = 0$, $Q \in L(P)$.

Example 1.4.3. P is hypoelliptic, that is, P satisfies (1.4.2), if and only if all elements of $L(P)$ are constants (of modulus one).

Example 1.4.4. If η is a simple characteristic of P, then the limits of $P_\xi/\tilde{P}(\xi)$ as $\xi \to \infty$ and $\xi/|\xi| \to \eta/|\eta|$ are of the form

$$a \sum_1^n P_m^{(j)}(\eta) D_j + b$$

where $a \geqq 0$ and $|a|^2 \Sigma |P_m^{(j)}(\eta)|^2 + |b|^2 = 1$. Thus we have a first order operator acting along the bicharacteristic corresponding to η.

The preceding example suggests an extension of the notion of bicharacteristic. If Q is a polynomial, we write

$$\Lambda(Q) = \{\eta \in \mathbf{R}^n; Q(\xi + t\eta) \equiv Q(\xi)\}$$

for the largest vector space in $\mathbf{R}^n$ along which Q is constant, and we introduce the annihilator

$$\Lambda'(Q) = \{x \in \mathbf{R}^n; < x, \eta > = 0, \ \eta \in \Lambda(Q)\},$$

which is the smallest subspace such that $Q(D)$ operates along $\Lambda'(Q)$. This means that $Q(D)u(0)$ is determined by the restriction of u to $\Lambda'(Q)$ and that $\Lambda'(Q)$ is the smallest subspace of $\mathbf{R}^n$ with this property. When $Q \in L(P)$ is not constant so that $dim\, \Lambda'(Q) > 0$, the planes parallel to $\Lambda'(Q)$ will be called *bicharacteristic spaces* for P. (These are the same for $P(D)$ and the adjoint $P(-D)$.) For every such plane B the equation $Q(D)u = 0$ obviously has solutions with supp $u = B$. Arguing along the lines familiar in geometrical optics one can make the heuristic arguments above precise and show that the equation $P(D)u = 0$ has a solution with sing supp $u = B$. This leads to

THEOREM 1.4.5. *If X is P-convex with respect to singular supports, it follows that the minimum principle is valid for d_X on all bicharacteristic spaces for P.*

When some $Q \in L(P)$ is non-elliptic as an operator in $A'(Q)$, this result can be improved (see e.g. Corollary 3.5 in Hörmander [7]). However, Theorem 1.4.6 below indicates that it may well be that the condition in Theorem 1.4.5 is sufficient if all $Q \in L(P)$ are elliptic. In this situation we see from Theorem 1.3.8 that the necessary condition in Theorem 1.4.5 means that X is Q-convex with respect to supports for every $Q \in L(P)$. It may perhaps be true in more general circumstances that P-convexity with respect to singular supports is equivalent to Q-convexity with respect to supports for all $Q \in L(P)$.

THEOREM 1.4.6. *X is P-convex with respect to singular supports if either of the following conditions is fulfilled:*

i) *$X \cap V$ is convex if V is any bicharacteristic space for P;*

ii) *All bicharacteristic spaces are 1-dimensional and d_X satisfies the minimum principle in all of them;*

iii) *All $Q \in L(P)$ are of order $\leqq 1$ and d_X satisfies the minimum principle in all bicharacteristic spaces.*

For the cases i) and ii) proofs are given in Hörmander [7]. They depend on modifications of the construction of fundamental solutions given in section 1.1 above. The proof of iii) will be given in section 1.5.

1.5. *Propagation of singularities for solutions of operators with first order localizations at infinity*

Let $P(D)$ be a differential operator such that every $Q \in L(P)$ is a first order operator. Since $P(D+\xi) = \Sigma P^{(\alpha)}(\xi) D^\alpha/\alpha\,!$ this means that we assume

$$(1.5.1) \qquad P^{(\alpha)}(\xi)/\tilde{P}(\xi) \to 0 \quad \text{when} \quad \xi \to \infty \quad \text{if} \quad |\alpha| > 1 .$$

This condition is analogous to the condition (1.4.2) for hypoellipticity, and it is fulfilled by any product of one hypoelliptic operator and one operator with simple characteristics. If $x \in \mathbf{R}^n$ we denote by B_x the closure of the set of bicharacteristic spaces for P containing x. Condition iii) in Theorem 1.4.6 clearly does not change if in addition to bicharacteristic spaces we consider limits of such spaces. (It may be appropriate to call such limits also bicharacteristic.) The last part of Theorem 1.4.6 is therefore a consequence of

THEOREM 1.5.1. *Let* $u \in \mathscr{D}'(X)$ *where* $X \subset \mathbf{R}^n$ *is an open set, and assume that* $P(D)u \in C^\infty(X)$. *If* $x \in \operatorname{sing\,supp} u$ *it follows that for some* $b \in B_x$ *the component of* $X \cap b$ *containing* x *is a subset of* sing supp u.

With $X_0 = X \setminus \operatorname{sing\,supp} u$ there is an equivalent statement which is more convenient in the proof:

THEOREM 1.5.2. *Let* $X_0 \subset X$ *be open,* $u \in \mathscr{D}'(X)$, $P(D)u \in C^\infty(X)$ *and* $u \in C^\infty(X_0)$. *If* $x \in X$ *and the component of* $X \cap b$ *containing* x *meets* X_0 *for every* $b \in B_x$, *it follows that* $u \in C^\infty$ *in a neighborhood of* x.

Since B_x is compact the hypothesis will still be fulfilled if X is replaced by a sufficiently large relatively compact subset. We may then assume without restriction that $u \in \mathscr{E}'(\mathbf{R}^n)$.

The first step in the proof is to localize the spectrum of u. Let ρ be any number with $0 < \rho < 1$. As in Hörmander [7] we can choose a partition of unity $1 = \Sigma_0^\infty \psi_j$ in $\mathbf{R}^n$ such that

i) $$0 \leqq \psi_j \in C_0^\infty$$

and

$$|\xi - \xi_j| < C|\xi_j|^\rho \; \textit{if} \; \xi \in \operatorname{supp} \psi_j\,; \; \psi_j(\xi) = 1 \; \textit{if} \; |\xi - \xi_j| < c|\xi_j|^\rho,$$

for some constants c, C and a sequence $\xi_j \in \mathbf{R}^n$.

ii) $$\sup |D^\alpha \psi_j| \leqq C_\alpha |\xi_j|^{-\rho|\alpha|}.$$

Note that i) implies that

$$\sum_0^\infty \xi_j^{n\rho - a} \leqq C \int_{|\xi|>1} |\xi|^{-a} d\xi < \infty \; \textit{if} \; a > n.$$

Condition ii) implies that for every positive integer N

(1.5.2) $$|\mathscr{F}^{-1}\psi_j(x)| < C_N |\xi_j|^{n\rho} (1 + |x||\xi_j|^\rho)^{-N}.$$

LEMMA 1.5.3. *If $u \in \mathscr{E}'(\mathbf{R}^n)$ is of order μ and $\hat{u}_j = \psi_j \hat{u}$, then*

$$\sup |u_j| \leqq C |\xi_j|^{\mu + n\rho} . \tag{1.5.3}$$

For an open set Y we have $u \in C^\infty(Y)$ if and only if for every compact set $K \subset Y$ and every positive integer N

$$\sup_{x \in K} |u_j(x)| < C_{N,K} |\xi_j|^{-N} . \tag{1.5.4}$$

Proof. (1.5.3) is obvious and so is (1.5.4) for every K if $u \in C_0^\infty(\mathbf{R}^n)$. In view of (1.5.2) it is also clear that (1.5.4) is valid outside supp u. Combination of these facts proves that (1.5.4) is valid if K does not meet sing supp u. On the other hand, assume that (1.5.4) is valid in a neighborhood of K. Since u is of exponential type at most $C|\xi_j|$ it follows from (1.5.3) that

$$|u_j(z)| \leqq C |\xi_j|^{\mu + n\rho} \exp(C|\xi_j||\operatorname{Im} z|), \quad z \in \mathbf{C}^n .$$

Hence $|u_j(z)| \leqq C |\xi_j|^{\mu + n\rho}$ when $|\operatorname{Im} z| < 1/|\xi_j|$. Using for example the three lines theorem (cf. John [1]) we conclude that $u_j(z) = 0(|\xi_j|^{-N})$ for every N in the set of points in $\mathbf{C}^n$ at distance at most $1/2n|\xi_j|$ from K. But then Cauchy's inequality shows that $D^\alpha u_j(x) = 0(|\xi_j|^{-N})$ for all α and N when $x \in K$, which proves that $\Sigma D^\alpha u_j(x)$ is uniformly convergent in K for every α. Hence $u \in C^\infty$ in the interior of K which proves the lemma.

We shall apply Lemma 1.5.3 to the distributions u and $f = P(D)u$ which occur in Theorem 1.5.2. Thus we define u_j and f_j by $\hat{u}_j = \psi_j \hat{u}$ and $\hat{f}_j = \psi_j \hat{f}$. Then we have (1.5.4) for compact subsets of X_0, and if u is replaced by f we have (1.5.4) for compact subsets of X. The equation $P(D)u = f$ implies that $P(D)u_j = f_j$.

The spectrum of u_j is concentrated near ξ_j so we introduce

$$v_j(x) = u_j(x) e^{-i\langle x, \xi_j \rangle}, \quad g_j(x) = f_j(x) e^{-i\langle x, \xi_j \rangle} / \tilde{P}(\xi_j) .$$

The equation $P(D)u_j = f_j$ then becomes

$$\tilde{P}(\xi_j)^{-1} P_{\xi_j}(D) v_j = g_j . \tag{1.5.5}$$

Here v_j and g_j have the properties stated above for u_j and f_j, and they are of exponential type $C|\xi_j|^\rho$ by the property i) of the partition of unity.

By Proposition 2.4 in Hörmander [7] we can for every j choose $Q_j \in L(P)$ so close to $P_{\xi_j}/\tilde{P}(\xi_j)$ that $P_{\xi_j}(D)/\tilde{P}(\xi_j) = Q_j(D) - R_j(D)$ where

$$\tilde{R}_j(0) \leqq C |\xi_j|^{-b} \tag{1.5.6}$$

for some $b > 0$. We rewrite (1.5.5) in the form

$$(1.5.5)' \qquad (Q_j(D) - R_j(D))\, v_j = g_j\,.$$

To take advantage of the fact that the coefficients of R_j are small we multiply both sides by $Q_j(D)^{k-1} + Q_j(D)^{k-2} R_j(D) + \ldots + R_j(D)^{k-1}$ and obtain

$$(1.5.7) \qquad Q_j(D)^k v_j = R_j(D)^k v_j + \sum_{\nu=1}^{k} Q_j(D)^{k-\nu} R_j(D)^{\nu-1} g_j\,.$$

The terms in the sum are $0(|\xi_j|^{-N})$ for all N on compact subsets of X. Since v_j satisfies (1.5.3) and is of exponential type $C\,|\,\xi_j\,|^\rho$, we have for every α by Bernstein's inequality

$$|\,D^\alpha v_j\,| < C_\alpha\,|\,\xi_j\,|^{a+|\alpha|\rho}$$

where we have written $a = \mu + n\rho$. Using (1.5.6) we therefore obtain

$$|\,R_j(D)^k v_j\,| < C_k\,|\,\xi_j\,|^{a+k(m\rho-b)}\,.$$

If we choose ρ so small that $m\rho < b$, the right hand side will decrease like any desired power of $1/|\,\xi_j\,|$ if k is large. To complete the proof of the theorem it is therefore sufficient to show that for solutions of an equation $Q(D)^k v = h$ where $Q \in L(P)$, h is small in X, v is bounded in X and small in X_0, it is true uniformly with respect to k and Q that v is small near the point x in Theorem 1.5.2. This is essentially a consequence of classical convexity theorems but the uniformity needed here forces us to reconsider these carefully.

1) Let $I \subset \mathbf{R}$ be an interval with 0 in its interior and let I_0 be another interval of positive length $\subset I$. Then there exist constants C and δ, $0 < \delta < 1$, such that

$$(1.5.8) \qquad |\,u(0)\,| \leqq C^k (\sup_{I_0} |u|)^\delta\ (\sup_{I} |u|)^{1-\delta} \quad if\ (d/dx - \lambda)^k u = 0\,.$$

Here C and δ depend on I and I_0 but are independent of k and the complex number λ. To prove (1.5.8) we note that $u(x) = e^{\lambda x} p(x)$ where p is a polynomial of degree $k - 1$. Assuming for example that $\operatorname{Re}\lambda \geqq 0$ we choose a closed interval $I_1 \subset I$ in the open positive x-axis. For suitable positive constants

$$\sup_{I_0} |\,u\,| \geqq e^{-c_0 \operatorname{Re}\lambda} \sup_{I_0} |\,p\,|\,, \quad \sup_{I} |\,u\,| \geqq e^{c_1 \operatorname{Re}\lambda} \sup_{I_1} |\,p\,|\,.$$

By classical inequalities of Tschebyscheff we have for some constant C

$$|p(0)| \leqq C^k \sup_{I_0} |p|, \quad |p(0)| \leqq C^k \sup_{I_1} |p|.$$

Hence we obtain (1.5.8) if $\delta c_0 \leqq (1-\delta)\, c_1$, that is, if $\delta \leqq c_1/(c_0+c_1)$.

2) Let $X_0 \subset X_1$ be open sets in **C** such that some point of X_0 is in the component of 0 in X_1. Then one can find compact sets $K_j \subset X_j$ and constants C, δ with $0 < \delta < 1$ such that

$$(1.5.9) \quad |u(0)| \leqq C^k (\sup_{K_0} |u|)^\delta (\sup_{K_1} |u|)^{1-\delta} \text{ if } (\partial/\partial \bar{z} - \lambda)^k u = 0.$$

Here C is independent of k and of λ. A substitution $u = ve^{i<x,\xi>}$ where $(i\xi_1 - \xi_2)/2 = \lambda$ and ξ is real reduces the proof to the case $\lambda = 0$. It is sufficient to prove that if $0 < r < r_1$, $0 < r_0 < r_1$ then

$$(1.5.10) \quad \sup_{|z|<r} |u(z)| \leqq C^k (\sup_{|z|<r_0} |u|)^\delta (\sup_{|z|<r_1} |u|)^{1-\delta} \text{ if } (\partial/\partial \bar{z})^k u = 0$$
$$\text{when } |z| < r_1,$$

for if we join 0 to a point in X_0 by a polygon, repeated use of (1.5.10) will yield (1.5.9). For $k = 1$ the inequality (1.5.10) is included in the three circles theorem of Hadamard. In the general case we note that

$$u(z) = \sum_0^{k-1} \bar{z}^j u_j(z)$$

where u_j is analytic. When $|z| = R < r_1$ we have $\bar{z} = R^2/z$ and therefore

$$\left| \sum_0^{k-1} R^{2j} z^{k-1-j} u_j(z) \right| \leqq r_1^{k-1} \sup_{|z|<r_1} |u(z)| \text{ when } |z| \leqq R < r_1.$$

If $|z| \leqq r_1' < r_1$ and R varies between r_1' and r_1 it follows from the classical estimates of Tschebyscheff for the coefficients of a polynomial (in R) that

$$\sup_{|z|<r_1'} |u_j(z)| \leqq C^k \sup_{|z|<r_1} |u(z)|.$$

A similar estimate is valid if we replace r_1 by r_0 and r_1' by a positive number $r_0' < r_0$, But this reduces the proof of (1.5.10) to the case $k = 1$ where as already pointed out the inequality follows from the three circles theorem of Hadamard.

We can now prove the main lemma. Let M be a family of first order differential operators $Q(D)$ with $\tilde{Q}(0) = 1$. Assume that $Q \in M$ implies $Q_\eta/\tilde{Q}(\eta) \in M$ for $\eta \in \mathbf{R}^n$ and that M is closed in *Pol* (1). Denote by B the closure of the set of all $\Lambda'(Q)$ with $Q \in M$.

LEMMA 1.5.4. *Assume that $X_0 \subset X$ are open sets in $\mathbf{R}^n$ with $0 \in X$ and assume that for every $b \in B$ the component of 0 in $b \cap X$ contains some point in X_0. Then one can find compact sets $K_0 \subset X_0$ and $K \subset X$ such that*

$$(1.5.11) \quad |u(0)| \leqq C^k (\sup_{K_0} |u| + N_k(u))^\delta (\sup_K |u| + N_k(u))^{1-\delta},$$

$$N_k(u) = \sum_{|\alpha| \leqq k+n+1} \sup_K k^{k-|\alpha|} |D^\alpha Q(D)^k u|,$$

if $u \in C^\infty(X)$, $Q \in M$, and k is a positive integer. The constants C and δ do not depend on u, Q or k.

Proof. We shall first verify (1.5.11) when $Q(D)^k u = 0$ in a neighborhood of a sufficiently large compact set $K \subset X$. When $Q(D)$ is any fixed first order operator with $\Lambda'(Q) \in B$ this case of (1.5.11) is contained in (1.5.8) and (1.5.9). When $dim\, \Lambda'(Q) = 1$ the same constants and compact sets can be used for all Q with $\Lambda'(Q)$ close to a fixed line in B so the compactness of S^{n-1} shows that we can use the same constant for all $Q \in M$ with $dim\, \Lambda'(Q) = 1$. When $dim\, \Lambda'(Q) = 2$ we first note as in the proof of (1.5.9) that Q may be replaced by a real translate which contains no term of order 0. Let $M_0 \subset M$ be the closure of the set of all $Q \in M$ with $dim\, \Lambda'(Q) = 2$ and $Q(0) = 0$, $\tilde{Q}(0) = 1$. It follows from (1.5.9) that (1.5.11) is valid when $Q(D)^k u = 0$ on a large compact subset of X, uniformly for all $Q \in M_0$ in a neighborhood of an element with $dim\, \Lambda'(Q) = 2$. The operators in M_0 near an element Q_0 with $dim\, \Lambda'(Q_0) = 1$ can after multiplication by a factor of modulus 1 be written

$$Q(D)u = \langle a, \operatorname{grad} u \rangle + i \langle b, \operatorname{grad} u \rangle$$

where a and b are real, a is orthogonal to b, $|a|^2 + |b|^2 = 1$ and a is close to a unit vector in $\Lambda'(Q_0)$. Introducing a and b as basis vectors in $\Lambda'(Q)$ we obtain the homogeneous case of (1.5.11) from (1.5.9) with constants and compact sets depending only on Q_0.

It remains to extend (1.5.11) to the inhomogeneous case. Let $f \in C_0^\infty(K_1)$ where $K_1 \subset\subset X$ is a neighborhood of the compact set K obtained in the proof for the homogeneous case. We wish to solve the equation

$$(1.5.12) \qquad Q(D)^k u = f$$

when $Q \in M$. Since $\tilde{Q}(0) = 1$ and $1 = \tilde{Q}(0) \leqq (1+|\xi|)\, \tilde{Q}(\xi)$ we have $\tilde{Q}(\xi) \geqq (1+|\xi|)^{-1}$. With the notations of (1.1.6) it follows that

$$|Q(\xi+\zeta)| \geqq C(1+|\xi|)^{-1} \quad \text{if} \quad \Phi(Q_\xi, \zeta) \neq 0\,.$$

Hence the solution of (1.5.12) given by

$$u(x) = (2\pi)^{-n} \int d\xi \int e^{i<x,\xi+\zeta>} \hat{f}(\xi+\zeta)\, Q(\xi+\zeta)^{-k}\, \Phi(Q_\xi, \zeta)\, d\lambda(\zeta)$$

has on every compact set an estimate of the form

$$(1.5.13) \qquad |u(x)| \leqq C^k \sum_{|\alpha| \leqq k+n+1} \sup |D^\alpha f| .$$

Here we have of course used the elementary and familiar fact that the right hand side of (1.5.13) bounds $(1+|\xi|)^{k+n+1}|\hat{f}(\xi)|$.

To prove (1.5.11) we just choose a function $\chi \in C_0^\infty(K_1)$ with $\chi = 1$ near K, $|D^\alpha \chi| \leqq (Ck)^{|\alpha|}$, $|\alpha| \leqq k+n+1$ (see e.g. Hörmander [11]), and solve as just explained the equation

$$Q(D)^k u_0 = f = \chi Q(D)^k u .$$

(Since we only need to know that (1.5.11) is valid for some constant depending on k instead of C^k it would be sufficient to use any fixed χ.) For u_0 we have the bound (1.5.13), and the estimate (1.5.11) is valid with u replaced by $u_1 = u - u_0$. Summing up, we obtain (1.5.11) with K_1 instead of K.

End of proof of Theorem 1.5.2. We may assume that the point x in the theorem is the origin. Then the hypotheses of Lemma 1.5.4 are fulfilled with $M = L(P)$. In view of the translation invariance of (1.5.11) it follows that if V is a compact connected neighborhood of 0 such that $K_0 + V$ and $K + V$ are contained in X_0 and X respectively, then

$$(1.5.11)' \quad \sup_V |v| \leqq C^k (\sup_{K_0+V} |v| + N)^\delta \, (\sup_{K+V} |v| + N)^{1-\delta}, \quad v \in C^\infty, Q \in L(P) ,$$

where we have written

$$N = \sum_{|\alpha| \leqq k+n+1} \sup_{K+V} k^{k-|\alpha|} |D^\alpha Q(D)^k v| .$$

We shall apply this estimate with $v = v_j$ and $Q = Q_j$ using (1.5.7). We recall that $v_j = 0(|\xi_j|^{-N})$ in $K_0 + V$ for every N and that a similar estimate is valid in $K + V$ for any derivative of the sum in (1.5.7). Furthermore, since $R_j(D)^k v_j$ is of exponential type $C|\xi_j|^\rho$ we obtain

$$\sum_{|\alpha| \leqq k+n+1} \sup_{K+V} |D^\alpha R_j(D)^k v_j| \leqq C_k |\xi_j|^{a_1 + k((m+1)\rho - b)}$$

where $a_1 = a + (n+1)\rho$. We choose ρ so small that $(m+1)\rho - - b < - b/2$. Then (1.5.11)′ gives for large enough k

$$\sup_V |v_j| \leqq C_k (|\xi_j|^{-kb/2})^\delta \, (|\xi_j|^{\mu+n\rho})^{1-\delta} .$$

Since $\delta > 0$ is independent of k we obtain by choosing k large that $v_j = O(|\xi_j|^{-N})$ on V for all N. In view of Lemma 1.5.3 it follows that $v \in C^\infty$ in a neighborhood of 0, which completes the proof of Theorem 1.5.2.

Remark. The importance of " Hölder estimates " for the study of propagation of singularities has been emphasized by John [1]. He proved results of the form (1.5.11) for a *fixed* Q which is elliptic as an operator in $\Lambda'(Q)$. However, no study has yet been made of the required uniformity in $Q \in L(P)$ for higher order elliptic operators Q.

A number of special cases of Theorem 1.5.1 occur in the literature; see Hörmander [1, section 8.8], Grušin [1], Hörmander [7]. The corresponding question has also been much studied for variable coefficients (see Chapter III) and so has the analogous question with C^∞ replaced by real analytic functions (and sometimes distributions replaced by hyperfunctions); see Andersson [1], Kawai [1], [2], Hörmander [11].

1.6. *General wave front sets*

Additional information can be obtained from the proof of Theorem 1.5.2 if one considers not only where in X that the sequence v_j is not $O(|\xi_j|^{-N})$ for all N as $j \to \infty$ but also for which subsequences of $\{\xi_j\}$ that this occurs. We shall now introduce some concepts which allow us to state such conclusions. The simplest and perhaps most natural one is the compactification of $\mathbf{R}^n$ by a sphere at infinity used by Sato [1, 2] and which we shall also consider in Chapters II and III in connection with operators with variable coefficients.

More generally, let $f : \mathbf{R}^n \to \mathbf{R}^N$ be a proper embedding of $\mathbf{R}^n$ in some bounded open set in $\mathbf{R}^N$. Explicitly this means that we assume that f is bounded, continuous and injective, and that the range of f is disjoint from the set of limit points of $f(\xi)$ as $\xi \to \infty$. The closure of $f(\mathbf{R}^n)$ is then a compactification of $\mathbf{R}^n$. We denote it by W and the subset of limit points as $\xi \to \infty$ by W_0. Identifying $\mathbf{R}^n$ with $f(\mathbf{R}^n)$ by means of the homeomorphism f we can write $W = W_0 \cup \mathbf{R}^n$ where the union is disjoint and $\mathbf{R}^n$ is a dense open subset.

We make the following important assumptions:

(i) f is semi-algebraic, that is, the graph of f is semi-algebraic;

(ii) $f(\xi+\eta) - f(\xi) \to 0$ as $\xi \to \infty$ if η is fixed in $\mathbf{R}^n$.

It is well known that (ii) must be valid uniformly when $|\eta|$ is bounded. In fact, if $\varepsilon > 0$ then

$$E_N = \{ \eta ; |f(\xi+\eta) - f(\xi)| < \varepsilon, |\xi| > N \}$$

has positive measure for sufficiently large N, and $E_N - E_N$ is then a neighborhood of 0. For η in this neighborhood we have $|f(\xi+\eta) - f(\xi)| < 2\varepsilon$ when $|\xi| > N + C$. In view of the assumed pointwise convergence we conclude that (ii) is in fact uniform when η is bounded. Using (i) and the Tarski-Seidenberg theorem (see e.g. the appendix in Hörmander [1]) we conclude that for a suitable K

$$|f(\xi+\eta) - f(\xi)| < \varepsilon \quad \text{if} \quad |\eta| < \varepsilon^{-1} \quad \text{and} \quad |\xi| \geqq \varepsilon^{-K}.$$

Writing $\delta = 1/K$ we have therefore proved that (i) and (ii) imply

$$|f(\xi+\eta) - f(\xi)| < |\xi|^{-\delta} \quad \text{if} \quad |\eta| < |\xi|^{\delta}. \tag{1.6.1}$$

Example 1.6.1. If $f(\xi) = \xi(1+|\xi|^2)^{-1/2}$ the compactification is the unit ball, and W_0 is the unit sphere.

All conditions on f are satisfied if we take the direct sum of this f with another f_1 satisfying (i) and (ii) only. For f_1 we may for example take any quotient $P/\tilde{Q}$ where Q is hypoelliptic and P is weaker than Q (see the proof of Theorem 4.1.6 in Hörmander [1]). Example 1.6.1 is also essentially of this form with $P(\xi) = 1 + |\xi|^2$. Semi-elliptic operators give other useful examples.

For distributions $v \in \mathscr{E}'(\mathbf{R}^n)$ we now introduce the set

$W(v) = W_0 \setminus \{ w \in W_0 ; \hat{v}(\xi)|\xi|^N$ is bounded for every N in a fixed neighborhood of w in $\mathbf{R}^n \cup W_0 \}$.

Note that if this set is empty, then $\hat{v}(\xi)$ is rapidly decreasing at infinity so $v \in C_0^\infty$.

LEMMA 1.6.2. *If $v \in \mathscr{E}'$ and $\varphi \in C_0^\infty$, then $W(\varphi v) \subset W(v)$.*

Proof. Assume that $w \notin W(v)$. This means that for some $\varepsilon > 0$ the Fourier transform $\hat{v}(\xi)$ is rapidly decreasing when $|f(\xi) - w| < \varepsilon$. We claim that the Fourier transform of $v_1 = \varphi v$ is also rapidly decreasing when $|f(\xi) - w| < \varepsilon/2$. Note that when $|f(\xi) - w| < \varepsilon/2$ and $|\xi|$ is large we have $|f(\xi+\eta) - w| < \varepsilon$ if $|\eta| < |\xi|^\delta$, by virtue of (1.6.1). Hence

$$|\hat{v}_1(\xi)| \leqq \int |\hat{v}(\xi-\eta)\hat{\varphi}(\eta)|\, d\eta \leqq C_N |\xi|^{-N} + \\ + C \int_{|\eta|>|\xi|^\delta} |\hat{v}(\xi-\eta)|\,|\hat{\varphi}(\eta)|\, d\eta$$

if $|f(\xi) - w| < \varepsilon/2$. In the last integral we estimate $|\hat{v}(\xi-\eta)|$ by

$C(1+|\xi|)^{\mu}(1+|\eta|)^{\mu}$ where μ is the order of v, and conclude that it is also $O(|\xi|^{-N})$ for every N. The proof is complete.

We can now define the wave front set:

Definition 1.6.3. If $u \in \mathscr{D}'(X)$ we denote by $WF(u)$ the complement in $X \times W_0$ of the set of all (x, w) such that for some $v \in \mathscr{E}'$ equal to u in a neighborhood of x the Fourier transform of v is rapidly decreasing in a neighborhood of w, that is, $w \notin W(v)$.

From the lemma it follows that the fiber of $WF(u)$ over x is the limit of $W(\varphi u)$ when the support of φ converges to x while $\varphi(x) \neq 0$. The projection in X of $WF(u)$ is sing supp u. In fact, it is trivially included in sing supp u. On the other hand, if x is not in the projection of $WF(u)$ it follows by the compactness of W_0 and Lemma 1.6.2 that $\varphi u \in C^{\infty}$ for some $\varphi \in C_0^{\infty}$ with $\varphi(x) \neq 0$. Thus we have proved:

THEOREM 1.6.4. *The projection in X of $WF(u)$ is equal to* sing supp u.

If F is any closed subset of $X \times W_0$ one can find $u \in C(X)$ with $WF(u) = F$. In fact, since $C_F = \{u \in C(X),\ WF(u) \subset F\}$ is a Fréchet space it suffices, in view of the closed graph theorem and Baire's theorem, to show that when $F_1 \subsetneq F_2$ the topologies in C_{F_1} and C_{F_2} are not identical. If $(x_0, w_0) \in F_2 \setminus F_1$ and $\xi_j \in \mathbf{R}^n$ is a sequence with $f(\xi_j) \to w_0$, this follows if we consider a sequence $u(x)\, e^{i<x,\xi_j>}$ where $u \in C_0^{\infty}$ has support close to x_0.

The results of section 1.5 can now be improved as follows. For every $w \in W_0$ we introduce the set $L_w(P)$ of all limits of $P_{\xi}/\tilde{P}(\xi)$ as $\xi \to w$. The proof of Theorem 1.5.2 gives the following refinement of Theorem 1.5.1:

THEOREM 1.6.5. *Let $u \in \mathscr{D}'(X)$ where X is an open set in $\mathbf{R}^n$, and let $P(D)u = f \in C^{\infty}(X)$. Assume that $L(P)$ consists of first order operators and let $B_{x,w}$ be the set of all limits of $\Lambda'(Q_j) + \{x\}$ with $Q_j \in L_{w_j}(P)$ and $w_j \to w$. If $(x, w) \in WF(u)$ it follows that for some $b \in B_{x,w}$ the component of $(X \cap b) \times w$ containing (x, w) is also in $WF(u)$.*

The result is particularly satisfactory if $B_{x,w}$ has a unique minimal element. (Note that Theorem 1.6.5 is then equivalent to its local form.) For example, if P is an operator with simple characteristics and W_0 is the unit sphere, then $B_{x,w}$ is empty except when w is a characteristic, and $B_{x,w}$ then consists of the corresponding bicharacteristic through x. (See example 1.4.4.) It would be interesting to know if for every operator P there is some compactification for which $B_{x,w}$ has a unique minimal element. It may be possible to obtain such results by arguments of the type used by Gabrielov [1] to prove that for every P the closed union of all $\Lambda'(Q)$, $Q \in L(P)$, is a semi-algebraic set of codimension at least one.

For other definitions of the wave front set we refer to Sato [1, 2], and Sato and Kashiwara [1] for the case of hyperfunctions relative to real analytic functions, and to Hörmander [11] for the case of Schwartz distributions relative to any Denjoy-Carleman class of functions which is closed under differentiation and contains the real analytic functions.

Chapter II

Some spaces of distributions and operators

2.1. *Pseudo-differential operators*

In Chapter I all results ultimately depended on the Fourier transformation. When the coefficients are variable we need to have some substitute. The simplest case occurs in the construction of fundamental solutions for *elliptic* operators with variable coefficients. Classically this was done by perturbation arguments (the E. E. Levi parametrix method, Korn's approximation). These ideas are now embedded in a more manageable and precise form in the theory of pseudo-differential operators.

Let us first note that for an elliptic operator $P(D)$ with constant coefficients of order m we have for some constant C,

$$|\xi|^m \leqq C\,|P(\xi)|\,, \quad |\xi| > C\,,$$

if ξ is real or belongs to a narrow cone in $\mathbf{C}^n$ containing $\mathbf{R}^n$. Apart from an integration over a compact set, which contributes an entire analytic term, the fundamental solution constructed in section 1.1 is therefore simply

$$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>}\, \chi(\xi)/P(\xi)\hat{f}(\xi)\, d\xi\,.$$

Here χ is a fixed C^∞ function which is 0 when $|\xi| < C$ and 1 for large $|\xi|$. Differentiation under the sign of integration gives, with E also denoting the distribution such that $Ef = E * f$,