Chapter III Pseudo-differential operators with non-singular characteristics

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respectively. If A is a pseudo-differential operator in X of order μ then the product AK is in $I^{m+\mu}(X \times Y, \Lambda)$ and the principal symbol is the product of the principal symbol of K (considered as living on Λ') by that of A lifted from $T^*(X)$ to Λ' by the projection $\Lambda' \to T^*(X)$. If we multiply to the right instead the result is the same except that we shall use the projection from Λ' to $T^*(Y)$. If A and B are pseudo-differential operators in X and in Y respectively and if $AK = KB$ we conclude that for the principal symbols a and b of A and B we must have

$$a(\chi(y, \eta)) = b(y, \eta) \tag{2.3.8}$$

if the principal symbol of K is not 0 at $(\chi(y, \eta), (y, -\eta))$. Conversely, (2.3.8) implies that $AK - KB$ is of lower order. We can therefore successively construct the symbol of B for a given A so that $AK - KB$ is of order $-\infty$, provided that the wave front set of A is concentrated near a point where K is elliptic. This argument often allows one to pass from one operator to another with principal symbol modified by a homogeneous canonical transformation. (See also Lemma 3.2.2 below.)

The operators in $I^m(X \times Y, \Lambda')$ can be described by means of the classical *generating function*: For any point $(x_0, \xi_0, y_0, \eta_0)$ in the graph of χ one can choose local coordinates in neighborhoods of x_0 and y_0 so that there is a function $\varphi(x, \eta)$ in a conical neighborhood of (x_0, η_0) which is homogeneous of degree 1 with respect to η, such that χ is given by $(\varphi'_\eta, \eta) \to$ $\to (x, \varphi'_x)$ and $\det \varphi''_{x\eta} \neq 0$. The elements A in $I^m(X \times Y, \Lambda)$ with wave front set close to $(x_0, \xi_0, y_0, -\eta_0)$ are then as operators of the form

$$Au(x) = (2\pi)^{-n} \int e^{i\varphi(x,\eta)} a(x, \eta) \hat{u}(\eta) \, d\eta, \quad a \in S^m(X \times \mathbf{R}^n),$$

when u is in C_0^∞ in a neighborhood of y_0 and x is in a neighborhood of x_0. The assertions made above are easy to prove directly from this representation.

Chapter III

PSEUDO-DIFFERENTIAL OPERATORS WITH NON-SINGULAR CHARACTERISTICS

3.1. *Preliminaries*

Throughout this chapter X will denote a C^∞ manifold (all manifolds are tacitly assumed countable at infinity) and P a properly supported pseudo-differential operator in X of order μ with homogeneous principal symbol p. This means that p is a complex valued C^∞ homogeneous function of degree μ on $T^*(X)\backslash 0$ and that for every local coordinate system the full symbol of P differs from p by a symbol in $S^{\mu-1}$. We shall also require that the characteristics are simple, that is,

$$dp(x, \xi) \neq 0 \quad \text{if} \quad (x, \xi) \in T^*(X)\backslash 0 \quad \text{and} \quad p(x, \xi) = 0 . \tag{3.1.1}$$

The purpose is to give analogues of the existence theorems stated in Chapter I for the case of differential operators with constant coefficients, in particular part iii) of Theorem 1.4.6 and the related Theorems 1.5.1 and 1.5.2. This will require further conditions on P which will be introduced later on.

We shall now recall some classical facts concerning the integration of the first order differential equation

$$p(x, \operatorname{grad} u) = 0 . \tag{3.1.2}$$

At first it will be assumed that p is real valued. If $u \in C^2(Y)$ for an open set $Y \subset X$ and if u is real valued, then $\Lambda = \{(x, \operatorname{grad} u(x)), x \in Y\}$ is a *section* of $T^*(X)$ over Y on which (the restriction of) the invariant symplectic form $\sigma_X = \Sigma d\xi_j \wedge dx_j$ vanishes. In fact, the pullback of σ_X to Y by the section is

$$d(\Sigma \partial u/\partial x_j dx_j) = d\,du = 0 .$$

Conversely, if we have a C^1 section Λ of $T^*(X)$ over Y on which σ_X vanishes, we can define Λ in local coordinates by $\xi = \xi(x)$, and $\partial \xi_j/\partial x_k - \partial \xi_k/\partial x_j = 0$ so $\xi = du$ for some function u in Y (determined up to an additive constant) if Y is simply connected. The (local) integration of (3.1.2) is therefore equivalent to finding a (local) section Λ of $T^*(X)$ such that

(i) $\qquad \sigma = 0$ on Λ

(ii) $\qquad p = 0$ on Λ.

In other words, Λ shall be a Lagrangean manifold (see section 2.3) contained in $p^{-1}(0)$ such that the projection $\Lambda \to X$ is a diffeomorphism. Locally the last condition means just that Λ is transversal to the fiber of the projection $T^*(X) \to X$. In the local theory one can therefore concentrate on (i) and (ii).

The symplectic form σ is a non-degenerate skew symmetric bilinear form on the tangent space of $T^*(X)$. That a manifold Λ is Lagrangean therefore means that at every point $\lambda \in \Lambda$ the tangent space $T_\lambda(\Lambda)$ is its own orthogonal complement with respect to σ. If (ii) is valid we have $dp = 0$

on $T_\lambda(\Lambda)$. The tangent vector H_p to $T^*(X)$ corresponding to the covector dp by the definition

$$< t, dp > \; = \sigma(t, H_p), t \in T(T^*(X)),$$

is therefore tangential to Λ. One calls H_p the Hamiltonian vector field defined by p. In terms of local coordinates x in X and the corresponding coordinates (x, ξ) in $T^*(X)$ the Hamiltonian vector is given by

$$H_p = \Sigma(\partial p/\partial \xi_j \partial/\partial x_j - \partial p/\partial x_j \partial/\partial \xi_j).$$

If q is another C^1 function on $T^*(X)$, then

$$H_p q = < H_p, dq > \; = \sigma(H_p, H_q) = -\sigma(H_q, H_p) = -H_q p,$$

and in local coordinates

$$H_p q = \{p, q\} = \Sigma(\partial p/\partial \xi_j \partial q/\partial x_j - \partial p/\partial x_j \partial q/\partial \xi_j).$$

$\{p, q\}$ is called the *Poisson bracket* of p and q. For later reference we note the Jacobi identity

$$(3.1.3) \qquad \{p, \{q, r\}\} + \{q, \{r, p\}\} + \{r, \{p, q\}\} = 0$$

or equivalently

$$H_{\{p,q\}} = H_p H_q - H_q H_p = [H_p, H_q].$$

For the proof we first observe that $[H_p, H_q]$ is a first order differential operator. This implies that (3.1.3) is independent of the second order derivatives of r, and similarly by the symmetry (3.1.3) is independent of the second order derivatives of p and q. But if p, q, r are all linear functions it is clear that all terms in (3.1.3) vanish so (3.1.3) must always be valid. From the Jacobi identity it follows that the (local) group of transformations defined by the vector field H_p is *canonical*, that is, it preserves the symplectic form. In fact, it suffices to note that if $q_1, \ldots, q_{2n}$ are symplectic coordinates at a point m and $H_p q_j$ = constant then these functions remain symplectic coordinates along the orbit of H_p through m since $H_p\{q_j, q_k\} = -$ $-\{q_k, \{p, q_j\}\} - \{q_j, \{q_k, p\}\} = 0$.

We now return to the Cauchy problem for (3.1.2). Let M be a hypersurface in X and u_0 a C^1 function with no critical point on M. We want to find u satisfying (3.1.2) and the Cauchy boundary condition $u = u_0$ on M. In addition $\xi_0 = \operatorname{grad} u(x_0)$ is prescribed for some $x_0 \in M$ in such a way that ξ_0 restricted to $T_{x_0}(M)$ is equal to grad u_0. We can then extend u_0 to a neighborhood of M so that grad $u_0 = \xi_0$ at x_0. If M is defined by the equation $\rho = 0$ we shall then have grad $u =$ grad $u_0 + t$ grad ρ on M, $t = 0$ at x_0, so on M the equation (3.1.2) becomes $p\,(x, \text{grad}\, u_0 + t\, \text{grad}\, \rho) = 0$. The derivative with respect to t when $t = 0$ and $x = x_0$ becomes $\{p, \rho\}\,(x_0, \xi_0)$. If we assume that H_p (or more precisely the projection $p'_\xi\, \partial/\partial x$ of H_p in $T(X)$) is *transversal* to M, it follows from the implicit function theorem that this equation has a unique solution in a neighborhood V of x_0. With $u_1 = u_0 + t\rho$, the Cauchy problem is now to find a Lagrangean manifold contained in $p^{-1}\,(0)$ and including

$$\Lambda_0 = \{(x, \text{grad}\, u_1(x)), x \in M_0 = M \cap V\}\,.$$

We have already seen that Λ must contain the integral curves of the vector field H_p starting in Λ_0 and by assumption these are transversal to Λ_0. It follows that there is a unique local solution of the Cauchy problem. In fact, the local manifold generated by integral curves of H_p through Λ_0 is Lagrangean at Λ_0 since σ vanishes on Λ_0 and $\sigma\,(t, H_p) = \langle t, dp \rangle = 0$ if $t \in T(\Lambda_0)$. The fact that Λ is invariant under the canonical transformations $\exp\,(tH_p)$ proves that Λ is Lagrangean everywhere.

When p or the Cauchy data are complex the preceding arguments break down and there is in general no solution unless p and the data are analytic (see section 3.3). However we always have an analogous result for formal power series solutions at a point, and this can be applied when the data are in C^∞ by considering the Taylor series expansions. We can say more if the vector field H_p happens to have an integral curve Γ with initial data (x_0, ξ_0), that is, if there exists a regular C^∞ curve $t \to (x\,(t), \xi\,(t)) \in T^*(X)$ with $(x\,(0), \xi\,(0)) = (x_0, \xi_0)$ and

$$0 \neq (dx/dt, d\xi/dt) = c\,(t)\,(p'_\xi, -p'_x)$$

for some complex valued function c. Apart from the parametrization such a curve is uniquely determined by (x_0, ξ_0) since it is an integral curve of any one of the vector fields $H_{\text{Re}\,p}$ and $H_{\text{Im}\,p}$ which is $\neq 0$. If such a curve Γ exists we can consider Taylor expansions on Γ instead. Even if the data on M are complex valued we then obtain a complex valued function u such that $\text{Im}\, u$ vanishes to the second order on Γ, grad $\text{Re}\, u(x\,(t)) = \xi\,(t)$, the restriction of u to M has a given Taylor expansion at x_0 and $p\,(x, \text{grad}\, u)$ vanishes of infinite order on Γ. The last statement makes sense although $p\,(x, \xi)$ is not defined for complex values of ξ, for the derivatives of $p\,(x, \text{grad}\, u)$ can still be computed formally on Γ.

For a more complete though less geometrical treatment of the topics discussed here we refer to Carathéodory [1]. Since for us the equation $p = 0$

is the characteristic equation of the operator P, we shall use the terminology *bicharacteristic strip* (resp. *curve*) for an integral curve of the Hamiltonian field H_p contained in $p^{-1}(0)$ (resp. the projection of such a curve in X). Note that whereas the bicharacteristic strip is non-degenerate or reduced to a point, the bicharacteristic curve may have a cusp. A simple classical example of this is given by the Tricomi equation for which $p(x, \xi) = x_2 \xi_1^2 + \xi_2^2$. With suitable normalization of the parameter the bicharacteristic strips are given by $x_1 = x_1^0 - 2(ct)^3/3$, $x_2 = -t^2c^2$, $\xi_1 = c \neq 0$, $\xi_2 = -tc^2$. The cusps of the bicharacteristic curves occur when $t = 0$. (Some authors use the term bicharacteristic strip for any integral curve of H_p and null bicharacteristic strip for those on which p vanishes.)

3.2. *Operators with real principal part*

Let P be a properly supported pseudo-differential operator of order μ in a manifold X and assume that P has a real and homogeneous principal part p satisfying (3.1.1). In this case rather complete results on the propagation of singularities and existence theorems in $\mathscr{D}'/C^\infty$ have been obtained by Duistermaat and Hörmander [1]. Complete proofs of all statements in this section are given there. The following result should be compared with Theorem 1.6.5.

THEOREM 3.2.1. *If $u \in \mathscr{D}'(X)$ and $Pu = f$ it follows that $WF(u) \setminus WF(f)$ is a subset of $p^{-1}(0)$ which is invariant under the flow defined by the Hamilton vector field H_p in $p^{-1}(0) \setminus WF(f)$.*

Proof. That $WF(u) \setminus WF(f) \subset p^{-1}(0)$ is precisely the second part of (2.2.2). To prove the other part of the theorem we consider a point $m \in WF(u) \setminus WF(f)$. If $H_p(m)$ has the radial direction the bicharacteristic curve through m is a ray, and since $WF(u)$ is conic there is nothing to prove then. Otherwise we can apply

LEMMA 3.2.2. *Let $m \in T^*(X) \setminus 0$, $p(m) = 0$, and assume that $H_p(m)$ does not have the radial direction. Then there exist Fourier integral operators $A \in I^{\mu_1}(X \times \mathbf{R}^n, \Gamma')$, $B \in I^{\mu_2}(\mathbf{R}^n \times X, (\Gamma^{-1})')$ such that*

(i) *Γ is a closed conic subset of the graph of a homogeneous canonical transformation χ from a conic neighborhood U of m onto a conic neighborhood V of a point $\chi(m) \in T^*(\mathbf{R}^n) \setminus 0$.*

(ii) $(m, \chi(m))$ *and* $(\chi(m), m)$ *are non-characteristic points for* A *and* B *respectively.*

(iii) μ_1 *and* μ_2 *are given numbers with* $\mu_2 + \mu + \mu_1 = 1$, *and the full symbol of the pseudo-differential operator* BPA *is equal to* ξ_n *on a conic neighborhood of* $\chi(m)$.

Proof. Multiplication of P by an elliptic pseudo-differential operator of order $1 - \mu$ reduces the proof to the case $\mu = 1$. The hypothesis on $H_p(m)$ then makes it possible to introduce a system of canonical coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ near m in $T^*(X)$, which are homogeneous of degree 0 and 1 respectively, so that $\xi_n = p$. This gives the canonical transformation χ. Choosing B and A with reciprocal principal symbols we obtain that BPA has the principal symbol ξ_n near $\chi(m)$. By successive choice of the terms of decreasing order in the symbols of B and A one can make the lower order terms in BPA vanish near $\chi(m)$.

End of proof of Theorem 3.2.1. With the notations of the lemma we also choose $B_1 \in I^{-\mu_1}(\mathbf{R}^n \times X, (\Gamma^{-1})')$ so that $m \notin WF(AB_1 - I)$. Then $v = B_1 u \in \mathcal{D}'(\mathbf{R}^n)$ and $\chi(m) \in WF(v)$ for against our assumption we would otherwise obtain $m \notin WF(u)$ since $u = (I - AB_1)u + Av$. (Here we are using Theorems 2.2.8 and 2.2.9.) Since

$$D_n v = (D_n - BPA)v + BP(AB_1 - I)u + BPu$$

we have $\chi(m) \notin WF(D_n v)$. Thus we have reduced the proof to the case of the operator D_n for which it follows by writing down a solution of the equation $D_n v = f$ explicitly.

Remark. Using only pseudo-differential operators, we shall prove a more general result in section 3.5 (see also Hörmander [13]).

In the opposite direction we have

THEOREM 3.2.3. *Let* $I \subset \mathbf{R}$ *be an open interval and* $\gamma : I \to T^*(X)\backslash 0$ *be a map defining a bicharacteristic strip for* P *which is injective even after composition with the projection to* $S^*(X)$. *Denote by* Γ *the closed conic hull of* $\gamma(I)$ *and by* Γ' *the limit points, that is, the intersection of the closed conic hull of* $\gamma(I \backslash I_0)$ *when* I_0 *runs over all compact intervals contained in* I. *For any* $\nu = 0, 1, 2, \ldots$ *one can then find* $u \in C^\nu(X)$ *such that* $WF(u) \backslash \Gamma' = \Gamma \backslash \Gamma'$ *and* $WF(Pu) \subset \Gamma'$.

Note that Γ' is empty precisely when γ defines a proper map from I into X. Then we have $Pu \in C^\infty(X)$ and $WF(u) = \Gamma$.

Proof. We shall just indicate a slightly weaker construction for a compact subinterval I_0 of I, but the passage to the statement above is quite easy from there. Assume for example that $0 \in I_0$. There is nothing to prove if $H_p(\gamma(0))$ has the radial direction so we exclude this case. We can then choose a $n-1$ dimensional conic submanifold N_0 of $N = p^{-1}(0)$ through $\gamma(0)$ such that $H_p(\gamma(0))$ is not a tangent of N_0 and the symplectic form vanishes in N_0. If $\Phi(t, n)$ denotes the solution of the Hamilton-Jacobi equations $d(x, \xi)/dt = H_p(x, \xi)$ at time t which is n at time 0, then it is clear that there is a closed conic neighborhood $V_0 \subset N_0$ of $\gamma(0)$ such that the map

$$I_0 \times V_0 \ni (t, n) \to \Phi(t, n)$$

is injective. Hence it defines a closed conic subset Λ of a Lagrangean manifold on which $p = 0$. (See the discussion of the Cauchy problem in section 3.1.) One can now choose $u \in I^k(X, \Lambda)$ with $WF(Pu)$ close to $\Phi((\partial I_0) \times V_0)$ so that the principal symbol of u has a given restriction to N_0 with support in a small conic neighborhood of $\gamma(0)$ in N_0. The crucial point is that for phase functions φ defining Λ locally we have $p(x, \varphi_x') = 0$ when $\varphi_\theta' = 0$. From this one concludes that to make the principal symbol of Pu vanish except at $\Phi(\partial I_0 \times V_0)$ means to solve differential equations along the bicharacteristics of P contained in Λ. (See the remarks on geometrical optics in the introduction.) As usual one can then successively determine terms of decreasing order in the symbol of u so that the symbol of Pu is of order $-\infty$ except at $\Phi((\partial I_0) \times V_0)$. If the order k is suitably chosen the desired properties are obtained. (To see that $WF(u)$ can be squeezed into Γ and not only a neighborhood one can either use functional analysis (see section 3 in Hörmander [7]) or more general symbols. A third possibility is indicated in the proof of Theorem 3.4.1 below.)

Remark. Zerner [1] and Hörmander [7] have given similar results which are weaker in that they are local and that they require H_p not to be a tangent to the fiber of $T^*(X)$ so that the bicharacteristic curve is regular. These constructions do not require the global definition of Fourier integral operators as the proof of Theorem 3.2.3 does.

We can now give an analogue of part iii) of Theorem 1.4.6.

THEOREM 3.2.4. *Assume that no complete bicharacteristic strip of P stays over a compact set in X. Then the following conditions are equivalent:*

a) *P defines a surjective map from $\mathscr{D}'(X)$ to $\mathscr{D}'(X)/C^\infty(X)$.*

b) *For every compact set $K \subset X$ there is another compact set $K' \subset X$, which can be taken empty when K is empty, such that $u \in \mathscr{E}'(X)$ and* sing supp ${}^tPu \subset K$ *implies* sing supp $u \subset K'$.

c) *For every compact set $K \subset X$ there is another compact set $K' \subset X$ such that any interval on a bicharacteristic curve with respect to P having endpoints in K must belong to K'.*

Proof. b) implies a) by Theorem 1.2.4. Assume that c) is fulfilled, and let $u \in \mathscr{E}'(X)$. If $m \in WF(u) \setminus WF(Pu)$ it follows from Theorem 3.2.1 that each bicharacteristic half strip through m must contain some point in $WF(Pu)$ unless it stays in $WF(u)$ and therefore over a compact set. However, if a bicharacteristic half strip stays over a compact set, then the bicharacteristic strip through any one of its limit points in the sphere bundle stays over this compact set in both directions which we have excluded by hypothesis. Hence m lies on an interval of a bicharacteristic strip with end points over K which proves that b) follows from c). By using a more precise version of Theorems 3.2.1 and 3.2.3 and an argument close to the proof of Theorem 3.6.3 in Hörmander [1] one shows that a) implies c).

Assuming still that P has no bicharacteristic strip which stays over a compact set in X, we set $N = p^{-1}(0) \subset T^*(X) \setminus 0$ and let $C \subset N \times N$ be the *bicharacteristic relation* of pairs of points in N which are on the same bicharacteristic strip. It is then easy to verify that C is a homogeneous canonical relation if and only if condition c) in Theorem 3.2.4 is fulfilled. Let C^+ (resp. C^-) be the subset of pairs (n_1, n_2) with n_1 on the forward (backward) bicharacteristic strip starting at n_2. Using the calculus of Fourier integral operators outlined in section 2.3 and Lemma 3.2.2 above one can prove

THEOREM 3.2.5. *Assume that no complete bicharacteristic strip of P stays over a compact set in X and that condition* c) *in Theorem* 3.2.4 *is fulfilled. Then there exist right parametrices E^+ and E^- for P, that is, operators such that $PE^+ - I$ and $PE^- - I$ have C^∞ kernels, with the following properties*[1]*:*

a) *$E^\pm$ are continuous linear maps from $H_{(s)} \cap \mathscr{E}'$ to $H_{(s+\mu-1)}$ for every s.*

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[1]) (Added in proof) In fact $E^\pm$ are also left parametrices and $E^\pm E^- \in I^{1/2-\mu}(X \times X, C')$. (See Duistermaat - Hörmander [1])

b) *$WF'(E^+)$ (resp. $WF'(E^-)$) is contained in $\Delta^* \cup C^+$ (resp. $\Delta^* \cup C^-$) where Δ^* is the diagonal in $T^*(X)\backslash 0 \times T^*(X)\backslash 0$.*

c) *Outside Δ^* the kernels of E^+ and E^- are in $I^{1/2-\mu}(X \times X, C')$.*

Condition b) *determines $E^\pm$ uniquely* mod C^∞.

A still better result, essentially due to Grušin [1] for operators with constant coefficients, can be obtained in the following way. Let A^+ and A^- be properly supported pseudo-differential operators with $A^+ + A^- = I$. With E^+ and E^- as in Theorem 5.3.7 we obtain a new parametrix E if we set

$$E = E^+A^+ + E^-A^- .$$

It will inherit the continuity properties of E^+ and E^- listed above, and

$$WF'(E) \subset \Delta^* \cup \{(m, n) \in C^\pm, n \in WF(A^\pm)\} .$$

Using operators with symbols satisfying (2.1.3)′ one can arrange that $WF(A^\pm) = F^\pm$ are any closed cones in $T^*(X)\backslash 0$ with union equal to $T^*(X)\backslash 0$. By condition c) in Theorem 3.2.4 one obtains for a suitable choice of F^+ and F^- a parametrix which can be extended to a continuous map from $H_{(s)}(X)$ to $H_{(s+\mu-1)}(X)$ for every s. This gives back part a) of Theorem 3.2.4 in a more constructive way.

We have only given global existence theorems here. However, local results follow immediately and they require only that no bicharacteristic strip for P stays forever over a fixed point in X. In the next section we shall discuss some more serious obstacles to local solvability which may occur when p is complex valued.

3.3. *Necessary conditions for local solvability and hypoellipticity*

We shall now allow the principal part p of the pseudo-differential operator P to be complex valued. That this leads to a drastic change of the situation discussed in section 3.2 was first realized by H. Lewy [1]. He found that the equation

$$(\partial/\partial x_1 + i\partial/\partial x_2 + 2i(x_1 + ix_2)\,\partial/\partial x_3)\,u = f$$

does not have a solution in any open set for suitably chosen $f \in C^\infty(\mathbf{R}^3)$. Starting from this example some necessary and some sufficient conditions for existence of (local) solutions were given by the author (see Hörmander [1, Chap. VI, VIII] and for the case of pseudo-differential operators Hörmander [3]). Mizohata [1] observed that for the equation

$$(\partial/\partial x_1 + i x_1{}^k \partial/\partial x_2)\, u = f \tag{3.3.1}$$

there is an existence theorem for even k but no solutions near the x_2 axis for suitable f if k is odd. With this example as starting point more precise conditions for local existence of solutions have been obtained by Nirenberg and Trèves [1], [2] (see also Trèves [2], [3]) and by Egorov [2], [3]. We shall discuss these results here in a somewhat more precise form made possible by the notion of wave front sets.

DEFINITION 3.3.1. The operator P is said to be solvable at $x_0 \in X$ if there is an open neighborhood V of x_0 such that for every $f \in C^\infty(X)$ one can find $u \in \mathscr{D}'(X)$ with $Pu = f$ in V.

Introducing a positive C^∞ density in X we can form the adjoint ${}^tP$ of P and write the equation $Pu = f$ in V as

$$< u, {}^tPv > \; = \; < f, v > , \quad v \in C_0^\infty(V)\,.$$

We may assume that $V \subset\subset X$. Solvability implies that the bilinear form

$$C^\infty(X) \times C_0^\infty(V) \ni (f, v) \to \; < f, v >$$

is separately continuous if for f we take the C^∞ topology and for v the weakest topology which makes the mapping $v \to {}^tPv \in C^\infty(X)$ continuous. Hence the form is continuous (Banach-Steinhaus), which means that for some semi-norms N_1, N_2 in $C^\infty(X)$

$$|< f, v >| \leqq C N_1(f)\, N_2({}^tPv), \; f \in C^\infty(X), v \in C_0^\infty(V)\,.$$

N_1 and N_2 are continuous semi-norms in $C^k(X)$ for some k. The estimate is clearly valid also for $f \in C^k(X)$, and an application of the Hahn-Banach theorem to the map

$${}^tPv \to \; < f, v >$$

shows that for every $f \in C^k(X)$ one can find $u \in \mathscr{E}'^k(X)$ so that $Pu = f$ in V. We have therefore proved

PROPOSITION 3.3.2. *If P is solvable at x_0, then there is a neighborhood V of x_0 and an integer k such that for every $f \in C^k(X)$ one can find $u \in \mathscr{E}'^k(X)$ with $Pu = f$ in V.*

To prove that P is not solvable at x_0 it is therefore sufficient to exhibit arbitrarily smooth functions f such that $Pu - f$ is not smooth near x_0 for any distribution u. This property has the advantage that it can be localized in the cotangent bundle as indicated in section 2.2:

DEFINITION 3.3.3. *If* $(x_0, \xi_0) \in T^*(X)\backslash 0$ *and* $f \in \mathscr{D}'(X)$, *we shall say that* $f \in P\mathscr{D}'(X)$ *at* (x_0, ξ_0) *if one can find* $u \in \mathscr{D}'(X)$ *so that* $(x_0, \xi_0) \notin WF(Pu-f)$. *We shall say that P is solvable at* (x_0, ξ_0) *if this is possible for every* f.

Solvability of P at a point $(x_0, \xi_0) \in T^*(X)\backslash 0$ is closely related to smoothness there of solutions of the adjoint equation $P^*u = f$ when f is smooth and $WF(u)$ is close to (x_0, ξ_0). Such existence and smoothness questions will therefore be studied simultaneously in what follows. To trace the origin of our arguments we first digress to discuss boundary problems for elliptic operators briefly.

Consider as an example the Laplace equation $\Delta u = 0$ in an open set $X \subset \mathbf{R}^n$ with a differential boundary condition $Bu = f$ on the smooth boundary ∂X. If u_0 is the restriction of u to ∂X, then u is the Poisson integral of u_0 and the boundary condition $Bu = f$ can be written as a pseudo-differential equation $\tilde{B}u_0 = f$ where the principal symbol of $\tilde{B}$ is easy to compute. In this way the study of *elliptic* boundary problems (see Agmon-Douglis-Nirenberg [1] or Hörmander [1, Chap. X]) can always be reduced to the study of an *elliptic* system of pseudo-differential operators on the compact manifold ∂X. The reduction is possible quite generally, however. In particular we can take $B = \partial/\partial v$ where v is a non-vanishing vector field on ∂X such that the equation $\langle v, N\rangle = 0$ defines a non-singular submanifold Y of ∂X, if N is the interior normal of ∂X. From the results related to Lewy's equation referred to above it follows that there is (local) solvability of the boundary problem if on Y the derivative of $\langle v, N\rangle$ in the direction v (which is tangential to ∂X on Y) is negative whereas there is a non-existence theorem if it is positive. For regularity of solutions the opposite signs are required. (See Borelli [1], Hörmander [3].) This strange result was explained by Egorov and Kondrat'ev [1] who found that in the two cases one should respectively introduce an additional boundary condition on Y or allow a discontinuity there. The problem then becomes well posed and solutions are smooth apart from a smooth jump. The proof of Egorov and Kondrat'ev attacked the boundary problem directly but their result can be translated to a property of a certain pseudo-differential operator which is elliptic outside a submanifold Y of codimension one. General theorems of this type have been proved by Eškin [1] and Sjöstrand [1].

Here we shall to a large extent follow Sjöstrand but will only deal with the situation corresponding to $\partial u/\partial v = 0$ and given restriction of u to Y.

Let us first consider the typical example given by equation (3.3.1) with $f = 0$. If it is possible to take Fourier transforms with respect to x_2, the equation becomes

$$(\partial/\partial x_1 - x_1{}^k \xi_2)\hat{u}\,(x_1, \xi_2) = 0$$

with the solution $\hat{u}\,(x_1, \xi_2) = C\,(\xi_2) \exp\,(\xi_2 x_1{}^{k+1}/(k+1))$. If k is *odd* we set for $v \in C_0^\infty\,(\mathbf{R})$

$$Ev\,(x) = (2\pi)^{-1} \int_{-\infty}^{0} \exp\,\big(\xi_2\,(ix_2 + x_1{}^{k+1}/(k+1))\big)\,v\,(\xi_2)\,d\xi_2 =$$
$$= (2\pi)^{-1} \iint_{\xi_2<0} \exp\,\big(\xi_2\,(i\,(x_2 - y) + x_1{}^{k+1}/(k+1))\big)\,v\,(y)\,dy d\xi_2\,.$$

From the results of section 2.3 it follows that E maps $C_0^\infty\,(\mathbf{R})$ to $C^\infty\,(\mathbf{R}^2)$ and $\mathscr{E}'\,(\mathbf{R})$ to $\mathscr{D}'\,(\mathbf{R}^2)$ continuously, and it is clear that $PEv = 0$ if $P = (\partial/\partial x_1 + ix_1{}^k \partial/\partial x_2)$. Let $\gamma : \mathbf{R} \ni x_2 \to (0, x_2)$ be the inclusion of the x_2-axis. Since the x_2-axis is non-characteristic with respect to P, it follows from (2.2.2) and Theorem 2.2.5 that the restriction $\gamma^* Ev\,(x_2)$ is defined, and clearly we have

$$\gamma^* Ev\,(x_2) = (2\pi)^{-1} \int_{-\infty}^{0} e^{ix_2\xi_2}\,\hat{v}\,(\xi_2)\,d\xi_2\,.$$

Using Theorem 2.3.1 we see that

$$WF'\,(E) = \{(x_1, \xi_1, x_2, \xi_2, y_2, \eta_2);\ x_1 = \xi_1 = 0,\ x_2 = y_2,\ \xi_2 = \eta_2 < 0\}.$$

For suitable choice of v we obtain a solution $u = Ev$ of $Pu = 0$ with $WF\,(u)$ equal to any closed subset of $F = \{(x_1, \xi_1, x_2, \xi_2);\ x_1 = \xi_1 = 0,\ \xi_2 < 0\}$ and conclude that P is not hypoelliptic. Moreover, if $u \in \mathscr{E}'$ and $P^*u = f$, then $E^*f = 0$ because $E^*P^* = (PE)^* = 0$. In case we only have $(x_0, \xi_0) \notin WF\,(P^*u - f)$ for some $u \in \mathscr{D}'$ we can still conclude that $WF'\,(E^*)\ (x_0, \xi_0) \notin WF\,(E^*f)$. For every point in F this is a non-trivial necessary condition in order that $f \in P\,\mathscr{D}'(X)$ at (x_0, ξ_0). (By studying the inhomogeneous equation $Pu = f$ Sjöstrand also obtains the sufficiency.)

Let us more generally consider a pseudo-differential operator such that the principal symbol in a local coordinate system with coordinates varying over $\mathbf{R}^n$ is of the form

$$p\,(x, \xi) = \xi_n + ix_n{}^k\,q\,(x, \xi) \tag{3.3.2}$$

when ξ is in a conic neighborhood of $\xi_0 = (\theta_0, 0) \neq 0$ and x is near $0 \in \mathbf{R}^n$.

PROPOSITION 3.3.4. *Let p be of the form* (3.3.2) *with k odd and* $\operatorname{Re} q(0, \xi_0) < 0$. *If B is a pseudo-differential operator in $\mathbf{R}^{n-1}$ with $WF(B)$ contained in a sufficiently small conic neighborhood of* $(0, \theta_0)$, *there exists a Fourier integral operator $E : C_0^\infty(\mathbf{R}^{n-1}) \to C_0^\infty(\mathbf{R}^n)$ with continuous extension from $\mathcal{E}'(\mathbf{R}^{n-1})$ to $\mathcal{E}'(\mathbf{R}^n)$ such that*

(i) *PE has a C^∞ kernel.*

(ii) $WF'(E) = \{(x', x_n, \xi', \xi_n; y', \eta'), x_n = \xi_n = 0,$
$(x', \xi') = (y', \eta') \in WF(B)\}$

(iii) $\gamma^* E = B$ *if* $\gamma(x') = (x', 0) \in \mathbf{R}^n, x' \in \mathbf{R}^{n-1}$

Proof. Let b be a symbol for B vanishing outside a small conic neighborhood of $(0, \theta_0)$. In order to have (iii) we wish to write E in the form

$$Ev(x) = (2\pi)^{1-n} \int e^{i\varphi(x,\theta)} a(x, \theta) \hat{v}(\theta) d\theta = \tag{3.3.3}$$
$$= (2\pi)^{1-n} \iint e^{i(\varphi(x,\theta) - <y',\theta>)} a(x, \theta) v(y') dy' d\theta$$

where

$$\varphi(x, \theta) = < x', \theta >, a(x, \theta) = b(x', \theta) \text{ when } x_n = 0 . \tag{3.3.4}$$

In order to obtain (i) the rules of geometrical optics require that one first solves the characteristic equation

$$\partial\varphi/\partial x_n + i x_n^k q(x, \partial\varphi/\partial x) = 0 \tag{3.3.5}$$

approximately with the initial data of (3.3.4). By the general remarks made in section 3.1 or directly by just computing what $\partial^j\varphi/\partial x_n^j$ must be when $x_n = 0$ for every j, we obtain a solution φ of infinite order when $x_n = 0$, and

$$\varphi(x, \theta) = < x', \theta > - i x_n^{k+1} q(x', 0, \theta, 0)/(k+1) + O(x_n^{k+2}) .$$

Note that, in a neighborhood of $(0, \theta_0)$ in which the support of a will lie,

$$\operatorname{Im} \varphi(x, \theta) \geqq c\, x_n^{k+1} \mid \theta \mid \tag{3.3.6}$$

for some $c > 0$, which gives (ii) in view of Theorem 2.3.1. Following the rules of geometrical optics (see also the parametrix construction in section 2.1) we determine successively the terms in an asymptotic series for a such that (i) is fulfilled. In doing so we can let P act under the integral sign in (3.3.3) and use the same formal expansion of $P(e^{i\varphi(x,\theta)} a(x, \theta))$ as if P were a differential operator (cf. Hörmander [3], Nirenberg-Trèves [2] and Hörmander [4, Theorem 2.6]).

We can now continue the argument precisely as in the example above. It follows that we can choose u with $Pu \in C^\infty$ and $WF(u)$ equal to any closed cone F in a sufficiently small neighborhood of $(0, \xi_0)$ in $p^{-1}(0)$. We can also choose f as smooth as we please so that f is not in $P^*\mathscr{D}'(X)$ at any point in F. Putting this conclusion in a form which is invariant under the equivalence used in Lemma 3.2.2 we shall obtain the main results of this section.

PROPOSITION 3.3.5. *Let N_j, $j = 1, 2, 3$, be the sets of all $m \in T^*(X)\backslash 0$ with $p(m) = 0$ having the following properties*

(N_1) *There exist Fourier integral operators A, B with the properties* (i), (ii) *in Lemma* 3.2.2 *such that the principal symbol of BPA satisfies the conditions in Proposition* 3.3.4 *at* $\chi(m)$.

(N_2) $H^I\{p_1, p_2\}(m) = 0$, $|I| < \mu$; $H^I\{p_1, p_2\}(m) = \lambda^I c$, $|I| = \mu$, *for some even integer* $\mu \geqq 0$, $(\lambda_1, \lambda_2) \in \mathbf{R}^2\backslash 0$, *and real* $c < 0$; *here we have written* $p = p_1 + ip_2$, *denoted by H^I any product of $|I|$ Hamiltonian first order operators H_{p_1} or H_{p_2} and by λ^I the corresponding product of λ_1 or λ_2. If $\mu \neq 0$ then* $\lambda_2 H_{p_1}(m) - \lambda_1 H_{p_2}(m) = 0$.

(N_3) *For some even integer $\mu \geqq 0$ and complex number z we have*

$$(\operatorname{Re} zH_p)^j \{\bar{p}, p\}(m)/2i = 0 \text{ for } j < \mu \text{ and } < 0 \text{ for } j = \mu .$$

Then the closures of N_1, N_2, N_3 in $T^*(X)\backslash 0$ are equal.

Proof. $N_1 \subset N_2$. Since (N_2) is invariant under canonical transformations and multiplication of p by a non-vanishing factor q (or even transformation of (p_1, p_2) by a matrix with positive determinant) it suffices to check (N_2) when $p_1(x, \xi) = \xi_n - x_n{}^k \operatorname{Im} q(x, \xi)$, $p_2(x, \xi) = x_n{}^k \operatorname{Re} q(x, \xi)$ and $\operatorname{Re} q < 0$. Then we have $\{p_1, p_2\} = kx_n{}^{k-1} \operatorname{Re} q(x, \xi) + O(x_n{}^k)$, $H_{p_1} - \partial/\partial x_n$ and H_{p_2} vanish when $x_n = 0$ if $k > 1$. Since $x_n = 0$ we obtain (N_2) with $\mu = k - 1$, $c = k! \operatorname{Re} q$ and $\lambda = (1, 0)$ if $k > 1$. That $N_2 \subset N_3$ is trivial. To show that N_3 is in the closure of N_1 it suffices to consider a point in N_3 such that $z = 1$, that is,

$$H_{p_1}{}^j p_2(m) = 0 \text{ for } j \leqq \mu, H_{p_1}{}^{\mu+1} p_2(m) < 0 .$$

Since $p_2(m) = 0$ it follows that $H_{p_1}(m)$ does not have the radial direction. According to Lemma 3.2.2 we can therefore choose Fourier integral operators A and B satisfying conditions (i), (ii) there so that the principal part of BA is real and the real part of the principal symbol of BPA is ξ_n near $\chi(m)$.

To economize notation we assume that already $p_1(\xi) = \xi_n$. Then $H_{p_1} = \partial/\partial x_n$ and our hypotheses are now that $m = (0; \theta_0, 0)$, $\partial^j/\partial x_n{}^j p_2(0; \theta_0, 0) = 0$ for $j \leqq \mu$ and < 0 for $j = \mu + 1$. Hence $p_2(0, x_n; \theta_0, 0)$ has the sign of $-x_n$ for small x_n. It follows that the equation $p_2(x', x_n; \xi', 0) = 0$ for (x', ξ') close to $(0, \theta_0)$ has at least one zero where p_2 for increasing x_n changes sign from plus to minus. If we choose such a zero close to m of minimum multiplicity k, necessarily odd, we may conclude from the implicit function theorem applied to $\partial^{k-1} p_2/\partial x_n{}^{k-1}$ that the zeros of p nearby are defined by $\xi_n = 0$ and an equation $x_n = r(x', \xi')$ with $r \in C^\infty$ homogeneous of degree 0 with respect to ξ'. Noting that the Poisson bracket $\{\xi_n, x_n - r(x', \xi')\}$ is 1 it is easy to add further canonical coordinates to ξ_n and $x_n - r(x', \xi')$ to obtain a homogeneous canonical transformation changing these functions to ξ_n and x_n. Implementing this by Fourier integral operators as in Lemma 3.2.2 again we see that at some point corresponding to a point arbitrarily close to m the transformed operator BPA will have a principal part of the form $\xi_n + iq_1$ where $q_1(x, \xi', 0) = x_n{}^k q(x, \xi')$ and $q < 0$. Thus the principal part can be written $\xi_n(1+is) + ix_n{}^k q$ where s is real. Multiplication by an elliptic operator with symbol $(1+is)^{-1}$ reduces it to the desired form and completes the proof.

Definition 3.3.6. The closure of any one of the sets N_1, N_2, N_3 in Proposition 3.3.5 will be denoted by $N_-(p)$, and we write $N_+(p) = N_-(\bar{p})$ which corresponds to changing the signs in the definition of N_1, N_2, N_3.

Note that in the case of differential operators the fact that $p(x, \xi) = (-1)^\mu p(x, -\xi)$ implies that $N_+(p)$ and $N_-(p)$ differ by multiplication with -1 in the fibers of $T^*(X)$. Thus they are simultaneously empty. This is not the case for pseudo-differential operators. For example, the study of the oblique derivative problem mentioned above leads to

$$p(x, \xi) = \xi_n + icx_n |\xi|$$

where $c \in \mathbf{R}\backslash 0$. Then $p = 0$ is equivalent to $x_n = \xi_n = 0$ and $\{Re\, p, Im\, p\} = c|\xi|$ has the sign of c there, so either N_+ or N_- is empty but not both.

From Propositions 3.3.4 and 3.3.5 we obtain by simple functional analysis:

Theorem 3.3.7. *Let F_+ and F_- be arbitrary closed cones contained in $N_+(p)$ and $N_-(p)$. For every $k > 0$ one can find $f \in C^k(X)$ with*

$WF(f) = F_+$ such that f is not in $P\mathscr{D}'(X)$ at any point in F_+. One can also find $u \in \mathscr{D}'(X)$ with $WF(u) = F_-$ and $Pu \in C^\infty(X)$.

The theorem shows that every (local) existence theorem must assume that $N_+(p) = \varnothing$ and that hypoellipticity requires that $N_-(p) = \varnothing$. The first statement is the necessary condition of Egorov, Nirenberg and Trèves referred to above.

In the notation of Proposition 3.3.5 Egorov's form of the condition $N_-(p) = \varnothing$ is $N_3 = \varnothing$. To arrive at the version of Nirenberg and Trèves we consider a point $m \in T^*(X)\backslash 0$ with $p(m) = 0$ and $d\,\mathrm{Re}\,p(m) \neq 0$. The equation $\mathrm{Re}\,p = 0$ defines a smooth hypersurface S containing m, and through each point in S there is an oriented integral curve of $H_{\mathrm{Re}p}$ which stays in S. Since in condition (N_2) we must have $\lambda_1 \neq 0$ if $\mu > 0$, it follows from (N_2) and (N_3) that $N_-(p) = \varnothing$ if and only if in a neighborhood of m in S the restriction of $\mathrm{Im}\,p$ to integral curves of $H_{\mathrm{Re}p}$ never has a zero of finite order where the sign changes from positive to negative. This is the condition of Nirenberg and Trèves. They conjectured that a necessary and sufficient condition for solvability at m of the adjoint (if H_p does not have the radial direction) is that such sign changes do not occur at any zeros (of finite or infinite order). A proof of the invariance of this condition under multiplication of p by a non-vanishing factor was given in Nirenberg-Trèves [2, appendix]. In fact, they discuss a semiglobal version of the same condition but the statements are not precise in this respect. Note that solvability of P at (x_0, ξ_0) for every $\xi_0 \neq 0$ does not imply solvability at x_0. An example is the differential operator in $\mathbf{R}^2$

$$P = x_1\,\partial/\partial x_2 - x_2\,\partial/\partial x_1$$

which in view of Lemma 3.2.2 is locally solvable at any point (x_0, ξ_0) but is obviously not solvable at 0. In Theorem 3.2.4 such behavior is ruled out by the assumption that bicharacteristic curves cannot lie in a compact set and similar conditions should be imposed in general.

3.4. *Further necessary conditions for hypoellipticity*

The standard definition of hypoellipticity which we have used throughout is that P is hypoelliptic if

$$\text{sing supp}\, u = \text{sing supp}\, Pu, u \in \mathscr{D}'(X). \tag{3.4.1}$$

This means that for every open set $Y \subset X$

(3.4.2) $$u \in \mathscr{D}'(X), Pu \in C^\infty(Y) \Rightarrow u \in C^\infty(Y).$$

For operators with variable coefficients this condition may be fulfilled for a fixed Y, for example $Y = X$, while (3.4.1) is not valid. For example, if $X = \{x \in \mathbf{R}^2, 1 < |x| < 2\}$ and $P = x_1\partial/\partial x_2 - x_2\partial/\partial x_1 + 1$ then P is not hypoelliptic but (3.4.2) is valid if $Y = X$. On the other hand, using the notion of wave front sets we can also consider a stronger property than (3.4.1)

(3.4.3) $$WF(u) = WF(Pu), u \in \mathscr{D}'(X).$$

Such operators will be called *strictly hypoelliptic* here. All hypoelliptic differential operators with constant coefficients as well as the hypoelliptic operators discussed in Hörmander [4] (see section 2.1) are strictly hypoelliptic. (It seems quite clear that if wave front sets had been considered some 15 to 20 years ago, then (3.4.3) rather than (3.4.1) would have been taken as definition of hypoelliptic operators.)

An operator $P \in L^\mu(X)$ is called *subelliptic* if for some $\delta > 0$ and real s

(3.4.4) $$u \in H_{(s)}(X) \cap \mathscr{E}'(X), Pu \in H_{(s+1-\mu)}(X) \Rightarrow u \in H_{(s+\delta)}(X).$$

Elliptic operators correspond to $\delta = 1$. From (3.4.4) it follows that we have a seemingly much stronger property: For any $t \in \mathbf{R}$

(3.4.5) $$u \in \mathscr{D}'(X), Pu \in H_{(t)} \text{ at } m \in T^*(X)\backslash 0 \Rightarrow u \in H_{(t+\mu-1+\delta)} \text{ at } m.$$

In particular, *subellipticity implies strict hypoellipticity*. To prove (3.4.5) we choose a real number r so that $u \in H_{(r)}$ at m. Assuming that $r \leqq t + \mu - 1$ we shall prove that $u \in H_{(r+\delta)}$ at m; by iteration this gives (3.4.5). Choose a pseudo-differential operator A of order $r - s$ which is non-characteristic at m so that $Au \in H_{(s)}(X) \cap \mathscr{E}'(X)$ and $APu \in H_{(t-r+s)}(X)$. We have

$$PAu = APu - [A, P]u.$$

Here $APu \in H_{(t-r+s)} \subset H_{(s+1-\mu)}$ and $[A, P]$ is of order $\leqq r - s + \mu - 1$ so $[A, P]u \in H_{(s-\mu+1)}$ also. It follows from (3.4.4) that $Au \in H_{(s+\delta)}(X)$, hence that $u \in H_{(r+\delta)}$ at m.

Subelliptic operators were characterized by Hörmander [3] for $\delta = 1/2$ by means of a localization method which is also valid for arbitrary $\delta > 0$ (see Hörmander [4]). In a series of papers Yu. V. Egorov has analyzed the localized estimates for arbitrary $\delta > 0$; their complexity increases very much as $\delta \to 0$. In Egorov [2] it was announced that (3.4.4) (or (3.4.5)) is valid if and only if $N_-(p) = \varnothing$ (see Definition 3.3.6) and

(3.4.6) $$H_p{}^j \bar{p}(m) = 0, 0 \leqq j \leqq \mu \Rightarrow \delta \leqq 1/(\mu+2).$$

Here we have used the notations in Proposition 3.3.5 and μ may be equal to 0. However, according to the lecture by Egorov at the International Congress in Nice there is a gap in his proof of sufficiency when $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$ are linearly dependent. (When they are linearly independent a proof has been given in Egorov [3] and another is easily obtained by combination of the results in Hörmander [3] and [5].)

In this section we shall derive other necessary conditions for hypoellipticity from constructions of solutions with small singularities. These are variants of Theorem 3.2.3. The first result is a more precise version of one due to Trèves [5], [7].

THEOREM 3.4.1. *Let I be an interval $\subset \mathbf{R}$ and $I \ni t \to \gamma(t) \in T^*(X)\backslash 0$ a bicharacteristic strip for P, that is, $0 \neq \gamma'(t)$ is proportional to $H_p(\gamma(t))$ for every $t \in I$. If I_0 is a sufficiently small neighborhood of a point $t_0 \in I$ and Γ (resp. Γ') is the closed conic hull of $\gamma(I_0)$ (resp. $\gamma(\partial I_0)$) one can for $\nu = 0, 1, 2, \ldots$ find $u \in C^\nu(X)$ so that $WF(u) = \Gamma$, $WF(Pu) \subset \Gamma'$.*

Proof. There is nothing to prove if $\gamma(t)$ has a constant projection on the cosphere bundle. Otherwise we can after an application of Lemma 3.2.2 assume that $p'_\xi(\gamma(t_0)) \neq 0$. Let $\gamma(t_0) = (x_0, \xi_0)$ and choose a function φ so that

(i) $\varphi(x) = \langle x - x_0, \xi_0 \rangle + i\,|x - x_0|^2$ in Σ where Σ is a plane in $\mathbf{R}^n$ through x_0 which is transversal to $p'_\xi(\gamma(t_0))$.

(ii) If $\gamma(t) = (x(t), \xi(t))$ then grad $\varphi(x(t)) = \xi(t))$ for t near t_0, and $p(x, \operatorname{grad} \varphi) = 0$ of infinite order on the bicharacteristic curve $\{x(t)\}$.

By the remarks on first order differential equations given in section 3.1 it is possible to choose φ locally with these properties. Since $\operatorname{Im} \varphi$ vanishes to the second order on $\{x(t)\}$ it follows from (i) that

(3.4.6) $$\operatorname{Im} \varphi(x) \geqq c d(x)^2$$

where $c > 0$ and $d(x)$ is the distance from x to the curve. One can now repeat the proof of Theorem 3.2.3 to obtain u in the form of a Fourier integral operator with phase function $\theta\varphi(x)$.

It seems difficult to improve Theorem 3.4.1 to a global result analogous to Theorem 3.2.3 as one would like to do in order to study (3.4.2) for a fixed Y. To do so we would first have to give a global definition of spaces of

Fourier integral operators which correspond locally to phase functions φ such as the one just constructed. Besides the curve $\gamma(t)$, the most important data contained in φ are the second order derivatives of φ along the curve. Let $V(t)$ be the tangent space of $T(T^*(X))$ at $\gamma(t)$ reduced modulo $\gamma'(t)$ and restricted to the orthogonal space of $\gamma'(t)$. Then $V(t)$ is symplectic, and if $V_{\mathbf{C}}(t)$ is the complexification, the Hamiltonian field H_p gives symplectic bijections $\chi_{st} : V_{\mathbf{C}}(t) \to V_{\mathbf{C}}(s)$. The Lagrangean plane defined in local coordinates by $\delta\xi = \varphi''_{xx}\,\delta x$ gives a Lagrangean plane $\lambda(t)$ in $V_{\mathbf{C}}(t)$ with $\chi_{st}\,\lambda(t) = \lambda(s)$. To have (3.4.6) we must require that $\lambda(t)$ is positive in the sense that

$$\operatorname{Im}\sigma(T, \overline{T}) > 0 \text{ if } 0 \neq T \in \lambda(t).$$

This condition is preserved by symplectic transformations which preserve the real spaces $V(t)$ but not by general complex symplectic transformations. Thus positivity of $\lambda(t)$ does not imply positivity of $\lambda(s)$. This is why we could make a global statement of Theorem 3.2.3 but not of Theorem 3.4.1. However, we have no examples which prove that this global difficulty is not merely due to the method of proof.

Next we consider a point $m \in p^{-1}(0)\backslash(N_+(p) \cup N_-(p))$ where $H_{\operatorname{Re}p}(m)$ and $H_{\operatorname{Im}p}(m)$ are linearly independent. Then $p^{-1}(0)$ is near m a manifold of codimension 2 on which $\{\operatorname{Re}p, \operatorname{Im}p\} = 0$; conversely, these conditions imply that $m \notin N_+(p) \cup N_-(p)$. By the Jacobi identity it follows that $[H_{\operatorname{Re}p}, H_{\operatorname{Im}p}] = H_{\{\operatorname{Re}p,\, \operatorname{Im}p\}}$ is a linear combination of $H_{\operatorname{Re}p}$ and $H_{\operatorname{Im}p}$ on $p^{-1}(0)$. In view of the Frobenius theorem we conclude that through m there passes a two dimensional local integral manifold of the vector fields $H_{\operatorname{Re}p}$, $H_{\operatorname{Im}p}$, contained in $p^{-1}(0)$ of course. This we call the bicharacteristic strip through m. Combination of the proof of Theorem 8.3 in Hörmander [7] with an analogue of Lemma 3.2.2 gives easily

THEOREM 3.4.2. *Let $m \in p^{-1}(0)\backslash(N_+(p) \cup N_-(p))$, and assume that $H_{\operatorname{Re}p}(m)$, $H_{\operatorname{Im}p}(m)$ and the radial direction at m are linearly independent. If V is a sufficiently small neighborhood of m in the two dimensional bicharacteristic strip through m and Γ(resp. Γ') is the cone generated by $\overline{V}$ (resp. ∂V), then one can for $\nu = 0, 1, \ldots$ find $u \in C^\nu(X)$ so that $WF(u) = \Gamma$, $WF(Pu) \subset \Gamma'$.*

It is easy to prove a global version of this result analogous to Theorem 3.2.3, at least when V is simply connected. (For more precise results see Duistermaat-Hörmander [1]).

When the radial direction lies in the bicharacteristic two plane it seems hard to give simple general results. However, the following theorem contains a case discussed by Trèves [5, 7]. For the sake of simplicity we assume that the symbol of P is an asymptotic sum of homogeneous terms.

THEOREM 3.4.3. *Let $\Lambda \subset p^{-1}(0)$ be a conic Lagrangean manifold and assume that on Λ the projection of H_p on the tangent space of $S^*(X)$ is proportional to a real vector and $\neq 0$. Let Γ be the cone generated by a finite solution interval of this vector field which is not a closed curve in $S^*(X)$, and let Γ' be generated by the end points of the interval. Then one can for $v = 0, 1, \ldots$ find $u \in C^v(Y)$ so that $WF(u) = \Gamma$ and $WF(Pu) \subset \Gamma'$.*

Note that H_p is tangential to Λ so the real vector field on $S^*(X)$ assumed to exist must be tangential to the submanifold of $S^*(X)$ induced by Λ. The proof of Theorem 3.4.3 is a repetition of that of Theorem 3.2.3 if one notes that for a homogeneous symbol differentiation in the radial direction is equivalent to multiplication by the degree. The first order differential equation in the direction H_p occurring in the recursive determination of the amplitude can therefore be reduced to a differential equation with real coefficients.

Assuming the conjecture stated at the end of section 3.3, Trèves [7] deduced from the preceding results necessary conditions for hypoellipticity of *differential* operators P with non-singular characteristics which were also proved to be sufficient. If P is such an operator, the necessary conditions are derived as follows:

a) By Theorem 3.3.7 we must have $N_-(p) = \varnothing$, hence $N_+(p) = N_-(p)' = \varnothing$.

b) By Theorem 3.4.2 the projection in $T(X)$ of H_p must have a real direction if $p = 0$. (If P is strictly hypoelliptic we conclude that H_p itself must have a real direction modulo the radial direction. In view of Theorem 3.4.3 we then obtain a contradiction if H_p does not have a real direction at some point.) Assuming from now on that $p'_\xi \neq 0$ when $p = 0$ we obtain, if $H_{\mathrm{Re}\,p}(m)$, $H_{\mathrm{Im}\,p}(m)$ are linearly independent for some m with $p(m) = 0$, that the projection $p^{-1}(0) \to X$ has rank $n - 1$ at every point in some neighborhood of m. The projection is therefore a hypersurface Y, defined by an equation $\rho(x) = 0$ with grad $\rho \neq 0$. Since ρ vanishes on $p^{-1}(0)$ near m it follows that H_ρ is a linear combination of $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$. Hence $p(m') = 0$ implies

$p(m' + tH_p(m')) = 0$ if t is small and m' is close to m. But p is a polynomial in the fibers so this must be an identity in t. Thus p must vanish in the normal bundle $N(Y)$ of Y, which is a Lagrangean manifold. On $N(Y)$ we also obtain that H_p is a linear combination of $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$ which means that the hypotheses of Theorem 3.4.3 are fulfilled so that P cannot be hypoelliptic. This contradiction shows that indeed H_p must be proportional to a real vector.

c) By Theorem 3.4.1 there cannot exist any one dimensional bicharacteristic strip for p. Hence it follows from b) that $\mathrm{Im}\,p$ cannot vanish on an interval of a bicharacteristic strip for $\mathrm{Re}\,p$.

d) Let $p(m) = 0$ and assume that $H_{\mathrm{Re}\,p}(m) \neq 0$. If the conjecture at the end of section 3.3 is true, it follows that on each bicharacteristic strip of $\mathrm{Re}\,p$ in a neighborhood of m the restriction of $\mathrm{Im}\,p$ is everywhere $\leqq 0$ or everywhere $\geqq 0$. Only one of the cases can occur for otherwise there would exist a bicharacteristic strip for $\mathrm{Re}\,p$ on which $\mathrm{Im}\,p$ vanishes, in contradiction with c). Hence we conclude that either $\mathrm{Im}\,p \geqq 0$ in a neighborhood of m when $\mathrm{Re}\,p = 0$, or else the opposite inequality is valid. Since we can choose $a \in C^\infty$ near m so that $a\,\mathrm{Re}\,p + \mathrm{Im}\,p$ is constant on a vector field transversal to $(\mathrm{Re}\,p)^{-1}(0)$, this means that m belongs to the set $N_U(p)$ introduced in

DEFINITION 3.4.4. We shall denote by $N_U(p)$ the set of all $m \in T^*(X)\backslash 0$ such that for some C^∞ function q in a neighborhood of m we have $q(m) \neq 0$ and $\mathrm{Im}\,qp \geqq 0$.

Naturally the function q can be chosen homogeneous. The set $N_U(p)$ is open and contains the complement of $p^{-1}(0)$. Only the intersection with $p^{-1}(0)$ is therefore interesting and it might have been more appropriate to introduce only this set in the definition. Note that $N_U(p) \cap N_+(p) = N_U(p) \cap N_-(p) = \varnothing$ for any p.

Modulo the truth of the conjecture at the end of section 3.3 it is therefore proved that if p is hypoelliptic and $p'_\xi \neq 0$ when $p = 0$ then $N_U(p) = T^*(X)\backslash 0$ and there is no one dimensional bicharacteristic strip for p (condition c) above). Conversely, Trèves [7] also proved that these conditions imply hypoellipticity. We shall give a proof in the following section where we also study the wave front set of solutions of $Pu = f$ in $N_U(p)$.

3.5. *Sufficient conditions for solvability and hypoellipticity*

Apart from the results of Hörmander [3] and Egorov [2] already referred to, all such conditions given so far in the literature include the assumption

$$N_{+}(p) \cup N_{-}(p) = \varnothing . \tag{3.5.1}$$

This is a necessary condition in the case of differential operators but not in general. (Cf. Definition 3.3.6 and Theorem 3.3.7.) When (3.5.1) is fulfilled, p is *real analytic*, and $p = 0$ implies $p'_\xi \neq 0$, Nirenberg and Trèves [2] have proved that P is solvable at every point. In fact, they showed that for every $x_0 \in X$ and $s \in \mathbf{R}$ there is an open neighborhood V of x_0 such that for every $f \in H_{(s)}(X)$ one can find $u \in H_{(s+\mu-1)}(X)$ with $Pu = f$ in V. The analyticity assumption is needed to give control of the changes of signs in say $\operatorname{Im} p$ when $\operatorname{Re} p = 0$. Unfortunately the proof which is based on an abstract version of the energy integral method does not seem to lead to information concerning the propagation of singularities. For this reason we content ourselves here with a reference to part II of Nirenberg-Trèves [2] and subsequent additions to appear in the same journal.

However, in $N_U(p)$ the situation is not too different from the real case studied in section 3.2. In fact, Trèves [7] has succeeded in extending the geometrical optics constructions to operators with $N_U(p) = T^*(X)\backslash 0$. The main point is that, although there may be no strict solutions to the characteristic and transport equations, it is possible to find sufficiently good approximate solutions. From his proof one can also obtain information on the wave front sets. We shall indicate a different approach here based on the energy integral method which gives a shorter though less constructive proof.

PROPOSITION 3.5.1. *Let $u \in \mathscr{D}'(X)$ and $Pu = f$, and consider a bicharacteristic strip $I \ni t \to \gamma(t) \in T^*(X)\backslash 0$ for* $\operatorname{Re} p$ *where $I = \{ t \in \mathbf{R}; t_1 \leqq \leqq t \leqq t_2 \}$. Assume that $\operatorname{Im} p \geqq 0$ in a neighborhood of $\gamma(I)$. If $\gamma(I) \cap \cup WF(f) = \varnothing$ and $\gamma(t_2) \notin WF(u)$, it follows that $\gamma(I) \cap WF(u) = \varnothing$. More precisely, if $f \in H_{(s)}$ at $\gamma(I)$ and $u \in H_{(s+\mu-1)}$ at $\gamma(t_2)$, then $u \in H_{(s+\mu-1)}$ at $\gamma(I)$.*

Proof. The assertion about $WF(u)$ follows from the last statement applied not only to $\gamma(I)$ but also to bicharacteristic strips for $\operatorname{Re} p$ nearby. In proving the last statement we may assume that $u \in H_{(s+\mu-3/2)}$ at $\gamma(I)$. It is convenient to assume that $\mu = 1$ which can be brought about by multiplication of P to the left by an elliptic operator of order $1 - \mu$. Choose a closed conic neighborhood Γ of $\gamma(I)$ such that $\operatorname{Im} p \geqq 0$ in a neighborhood of Γ, $f \in H_{(s)}$ and $u \in H_{(s-1/2)}$ in Γ. It is clearly enough to prove Proposition 3.5.1 locally so we may assume that Γ has a compact projection in a coordinate patch which is identified with $\mathbf{R}^n$ and that $u \in \mathscr{E}'(\mathbf{R}^n)$.

Let $M \subset S^{s-1}(X \times \mathbf{R}^n)$ be a bounded subset of $S^s(X \times \mathbf{R}^n)$ which consists only of real valued functions with support in Γ. (We shall make an explicit choice of M later where the closure in S^s (in a weak topology) can contain symbols of order s.) With $c \in M$ we put $C = c(x, D)$ and form

$$(3.5.2) \qquad (Cf, Cu) = (CPu, Cu) = (PCu, Cu) + ([C, P]u, Cu).$$

Here (,) denotes the usual sesquilinear scalar product. Write $P = A + iB$ with A and B self-adjoint, that is, $A = (P+P^*)/2$, $B = (P-P^*)/2i$. The principal symbols a and b of A and B are $\operatorname{Re} p$ and $\operatorname{Im} p$ respectively. Taking the imaginary part of (3.5.2) we obtain

$$(3.5.3) \qquad \operatorname{Im}(Cf, Cu) = (BCu, Cu) + \operatorname{Re}([C, B]u, Cu) + \\ + \operatorname{Im}([C, A]u, Cu).$$

We can write $B = B_0 + B_1$ where the principal symbol of B_0 is non-negative everywhere and $WF(B_1)$ does not meet Γ. By a well known improvement of Gårding's inequality (Hörmander [3, Theorem 1.3.3]; see also Lax-Nirenberg [1], Kumano-go [1], Vaillancourt [1], and for a still more precise result Melin [1]) we have

$$(3.5.4) \qquad \operatorname{Re}(B_0 v, v) \geqq -C_1 \|v\|_{(0)}^2, v \in C_0^\infty,$$

where $\| \ \|_{(0)}$ is the norm in $L^2 = H_{(0)}$. (We use here the more restrictive definition of $H_{(s)}(\mathbf{R}^n)$ as $(1-\Delta)^{-s/2} L^2(\mathbf{R}^n)$.) Since $B_1 C$ is of order $-\infty$ we obtain with a constant C_2 depending on u but not on C

$$(3.5.5) \qquad (BCu, Cu) \geqq -C_1 \|Cu\|_{(0)}^2 - C_2.$$

Next we note that the symbol of $C^*[C, B]$ is $ic\{b, c\} = i\{b, c^2\}/2$ apart from an error which belongs to a bounded set in S^{2s-1}. Since $\{b, c^2\}$ is real valued it follows that the symbol of the sum of $C^*[C, B]$ and its adjoint is in a bounded set in S^{2s-1}, which shows that with another C_2 depending on u

$$(3.5.6) \qquad \operatorname{Re}([C, B]u, Cu) \geqq -C_2.$$

In the same way we obtain

(3.5.7) $\qquad 2\,\mathrm{Im}([C, A]\,u, Cu) \geqq \mathrm{Re}(\{a, c^2\}(x, D)\,u, u) - C_2\,.$

Summing up (3.5.3)-(3.5.7) we obtain with still another C_2

(3.5.8) $\qquad \mathrm{Re}(e(x, D)\,u, u) \leqq \| Cf \|_{(0)}{}^2 + C_2, C \in M\,,$

where

(3.5.9) $\qquad e(x, \xi) = \{a, c^2\}(x, \xi) - (2C_1 + 1)\,c(x, \xi)^2\,.$

Clearly $\| Cf \|_{(0)}$ is bounded when $C \in M$. Note that while C_2 and this bound may depend on M, the constant C_1 comes from (3.5.5) and is completely independent of the choice of M.

We may assume that the map from I to the cosphere bundle defined by γ is injective. Let Γ_0 be an open conic neighborhood of $\gamma(t_2)$ where $u \in H_{(s)}$ and choose a non-negative C^∞ homogeneous function c of degree s with support in Γ such that $\{a, c^2\} = H_{\mathrm{Re}\,p}\,c^2 \geqq 0$ in $\Gamma \backslash \Gamma_0$ with strict inequality in $\gamma(I) \backslash \Gamma_0$. That this is possible is seen immediately if we first define $c(x, \xi)$ for $|\xi| = 1$ using a norm in $T^*(X)$ which is constant on the integral curves of $H_{\mathrm{Re}\,p}$. Also choose C^∞ functions a_0 and a_1 homogeneous of degree 0 and 1 respectively so that $H_a a_0 = 1$, $H_a a_1 = 0$ and a_1 is different from 0 in the support of c. This is also possible if the support of c is a sufficiently small neighborhood of $\gamma(I)$. Now M will consist of the functions

$$c_{\lambda,\varepsilon} = c\,e^{\lambda a_0}(1 + \varepsilon^2 a_1{}^2)^{-1/2}, 0 < \varepsilon \leqq 1\,,$$

where λ is *fixed* $\geqq C_1 + 1$. If c is replaced by $c_{\lambda,\varepsilon}$ the function e in (3.5.9) becomes

$$e_{\lambda,\varepsilon} = (\{a, c^2\} + (2\lambda - 2C_1 - 1)\,c^2)\,e^{2\lambda a_0}(1 + \varepsilon^2 a_1{}^2)^{-1}\,.$$

Since $e_{\lambda,\varepsilon} \geqq 0$ outside Γ_0 with strict inequality on $\gamma(I) \backslash \Gamma_0$ we can choose a non-negative homogeneous function r of degree s which is positive on $\gamma(I)$, and a real valued homogeneous function q of order s with support in Γ_0, thus $q(x, D)\,u \in L^2$, such that

(3.5.10) $\qquad r^2 \leqq (\{a, c^2\} + (2\lambda - 2C_1 - 1)\,c^2)\,e^{2\lambda a_0} + q^2\,.$

Let $r_\varepsilon = r(1 + \varepsilon^2 a_1^2)^{-1/2}$ and $q_\varepsilon = q(1 + \varepsilon^2 a_1{}^2)^{-1/2}$. An application of (3.5.4) to the operator with principal symbol equal to the difference of the two sides in (3.5.10) multiplied by $|\xi|^{1-2s}$ leads to the estimate

$$\| r_\varepsilon(x, D)\,u \|_{(0)}{}^2 \leqq \mathrm{Re}(e_{\lambda,\varepsilon}(x, D)\,u, u) + \| q_\varepsilon(x, D)\,u \|_{(0)}{}^2 + C_3$$

since $u \in H_{(s-1/2)}$ in Γ. (Here we rely on the uniformity of (3.5.4) when the symbol of B_0 is bounded in S^1.) In view of (3.5.8) we conclude that

$\| r_\varepsilon(x, D) u \|_{(0)}$ is bounded when $\varepsilon \to 0$, which proves that the limit $r(x, D) u$ of $r_\varepsilon(x, D) u$ in $\mathscr{D}'$ must belong to L^2. Hence $u \in H_{(s)}$ at $\gamma(I)$, which proves the proposition.

Another way of stating the proposition is that if $\gamma(I) \cap WF(f) = \varnothing$ and $\gamma(t_1) \in WF(u)$, then $\gamma(I) \subset WF(u)$. In view of Theorem 2.2.2 it follows that $\gamma(I) \subset p^{-1}(0)$, which implies that $H_{\operatorname{Im} p} = 0$ on $\gamma(I)$ since $\operatorname{Im} p \geqq 0$. Thus $\gamma(I)$ is a bicharacteristic strip for p. This gives the following extension of a result of Trèves [7] mentioned above:

THEOREM 3.5.2. *If Γ is an open cone $\subset N_U(p)$ containing no bicharacteristic strip for p, then*

$$WF(Pu) \cap \Gamma = WF(u) \cap \Gamma, u \in \mathscr{D}'(X). \tag{3.5.10}$$

If $\Gamma \supset p^{-1}(0)$ it follows that P is strictly hypoelliptic.

We can also obtain conclusions concerning the global existence of solutions and the global regularity question (3.4.2). To state them we first have to discuss the orientation of the Hamilton field $H_p(m)$ when $m \in N_U(p) \cap p^{-1}(0)$. According to Definition 3.4.4 we can choose q so that $q(m) \neq 0$ and $\operatorname{Im} qp \geqq 0$ near m. With $p_1 = qp$ we have then $d \operatorname{Re} p_1(m) \neq$ $\neq 0$, $d \operatorname{Im} p_1(m) = 0$. If for another function r with $r(m) \neq 0$ we have $\operatorname{Im} rp_1 \geqq 0$ near m, then $r(m)$ is either positive or negative. In the latter case we obtain $\operatorname{Im} p_1 \leqq 0$ near m when $\operatorname{Re} p_1 = 0$, and since $\operatorname{Im} p_1 \geqq 0$ it follows that $\operatorname{Im} p_1 = 0$ near m when $\operatorname{Re} p_1 = 0$. Hence $\operatorname{Im} p_1 = s \operatorname{Re} p_1$ for some smooth s, which means that $p_1 = (1+is) \operatorname{Re} p_1$ is real apart from a non-vanishing factor. If p is not of this special form we conclude that $r(m) > 0$, hence that $H_{rp_1}(m) = r(m) H_{p_1}(m)$ has the same direction as $H_{p_1}(m)$.

DEFINITION 3.5.3. By $N_R(p)$ we denote the set of all $m \in T^*(X)\backslash 0$ such that there is a C^∞ function q in a neighborhood of m with $q(m) \neq 0$ and qp real.

$N_R(p)$ is of course an open subset of $N_U(p)$ containing the complement of $p^{-1}(0)$. In $N_R(p) \cap p^{-1}(0)$ there is no natural way of choosing a complex number z such that zH_p is real, but if $m \in N_U(p)\backslash N_R(p)$ we choose as positive the direction of $q(m) H_p(m)$ when $q(m) \neq 0$ and $\operatorname{Im} qp \geqq 0$ in a neighborhood of m. The arguments preceding Definition 3.5.3 proved precisely that this definition is unique.

In $N_R(p)$ we have the situation studied in section 3.2. However, the orientation of the Hamiltonian field in $N_U(p)\backslash N_R(p)$ enters the analogue of Theorem 3.2.1 there.

THEOREM 3.5.4. *Let $u \in \mathscr{D}'(X)$ and $Pu = f$.*
If $m \in (WF(u)\backslash WF(f)) \cap N_U(p)$, then there exists a bicharacteristic strip $I \ni t \to \gamma(t) \in N_U(p)\backslash WF(f)$ for p with $m \in \gamma(I) \subset WF(u)$ such that I is a (finite) interval on $\mathbf{R}$ and, if t_0 is a boundary point of I,

(i) *$\gamma(t_0) \in N_U(p)\backslash N_R(p)$ and the positive direction of $H_p(\gamma(t_0))$ points towards $\gamma(I)$ if $t_0 \in I$.*

(ii) *$\gamma(t)$ does not converge to a limit in $N_U(p)\backslash WF(f)$ as $I \ni t \to t_0$ if $t_0 \notin I$.*

The proof follows from Proposition 3.5.1.

We can now give a partial extension of Theorem 3.2.4. Assume that $N_U(p) = T^*(X)\backslash 0$. We shall say that a curve $I \ni t \to \gamma(t) \in p^{-1}(0)$ is a complete bicharacteristic strip for p if I is a finite interval in $\mathbf{R}$ and

(i) $d\gamma/dt$ is proportional to $H_p(\gamma(t))$, $t \in I$,

(ii) $\gamma(t_0) \in N_U(p)\backslash N_R(p)$ and the positive direction of $H_p(\gamma(t_0))$ points towards $\gamma(I)$ if t_0 is a boundary point of I belonging to I.

(iii) $\gamma(t)$ does not converge to a limit in $N_U(p)$ as $I \ni t \to t_0$ if $t_0 \notin I$.

THEOREM 3.5.5. *Assume that $N_U(p) = T^*(X)\backslash 0$ and that no complete bicharacteristic strip for $\bar{p}$ stays over a compact set in X. Every $u \in \mathscr{E}'(X)$ with $P^*u \in C^\infty(X)$ is then in $C_0^\infty(X)$, which implies that the equation $Pu = f$ can be solved in a neighborhood of any compact set $K \subset X$ when f is orthogonal to the finite dimensional vector space of functions $v \in C_0^\infty(K)$ with $P^*v = 0$. The map from $\mathscr{D}'(X)$ to $\mathscr{D}'(X)\backslash C^\infty(X)$ defined by P is surjective if in addition for every compact set $K \subset X$ there is another compact set $K' \subset X$ such that K' contains the projection of any compact interval I on a complete bicharacteristic strip J for $\bar{p}$ with the projection of the boundary of I relative to J contained in K.*

The proof is a repetition of part of the proof of Theorem 3.2.4 with Theorem 3.2.1 replaced by Theorem 3.5.4.