# 5. Effective methods. Baker's Theorem

Put $N_v = 1$ or $N_v = 2$ if $1 \leqq v \leqq r$ or if $r + 1 \leqq v \leqq r + s$, respectively; $N_v$ was defined above when $v$ is a prime ideal. Put $\| \alpha \|_v = |\alpha|_v^{N_v}$. It is clear that the definition of $| \alpha |_v$ and of $\| \alpha \|_v$ can be extended to $\alpha \in K_v$. The product formula

$$\prod_{v \in \Omega} \| \alpha \|_v = 1$$

holds for every non-zero $\alpha$ in $K$. One can show that

$$c_1(K) H_K(\beta) \leqq \prod_{v \in \Omega} \max(1, \| \beta \|_v) \leqq c_2(K) H_K(\beta) . \tag{4.9}$$

THEOREM 4G. *Let $S$ be a finite subset of $\Omega$. For each $v \in S$, let $\alpha_v$ be an element of $K_v$ which is algebraic over $K$. For $\beta \in K$ put*

$$g(\beta) = \prod_{v \in S} \min(1, \| \alpha_v - \beta \|_v) .$$

*Then for every $\delta > 0$ there are only finitely many $\beta \in K$ with*

$$g(\beta) < H_K(\beta)^{-2-\delta} .$$

A more general version of this theorem may be found in Mahler ((1961), Appendix C, Assertion (2,I)). See also Lang ((1962), ch. 6). In its present form Theorem 4G does not contain Theorem 4F, but Mahler's generalization of it does. The case of Theorem 4G when $K$ is the field of rationals is due to Ridout (1958). It may be seen that Mahler's Theorem 4C is equivalent with the case of Theorem 4G when $S = \{1, 2, ..., r + s\}$, i.e. when we are considering only Archimedean valuations. Lang and Ridout also gave $p$-adic versions of Theorem 2C and thus sharpened Theorem 4E. Like Roth's Theorem, the results discussed here do not permit to give an estimate for the " size " (say $H(\beta)$) of the solutions, and hence they are non-effective. For weaker but effective $p$-adic results see Theorem 5B, 5D and 5E. For an effective weaker version of Theorem 4G see Sprindžuk (1970b, 1971a).

## 5. EFFECTIVE METHODS. BAKER'S THEOREM

**5.1.** All the results obtained by the method of Thue, Siegel and Roth share the disadvantage that they are non-effective. Although they show that certain inequalities and equations have only finitely many integer solutions, they do not give bounds for the size of the solutions and hence give no method to compute all the solutions.

Effective bounds, which however do not imply Roth's Theorem, and which do not imply Thue's or Siegel's Theorem unless $\alpha$ is of a special type, were given by Baker. He first used hypergeometric series (see also Siegel (1937)) to deal with algebraic numbers of the type $\alpha = \sqrt[d]{a/b}$. He showed (1964a) that if $\kappa > 2$, $d \geqq 3$ and if $a, b$ are integers with $b > 0$, $a > (a-b)^{c_1} c_2$ where $c_i = c_i(\kappa, d)$ $(i=1, 2)$, then all rational numbers $p/q$ with $q > 0$ satisfy

$$\left| \sqrt[d]{\frac{a}{b}} - \frac{p}{q} \right| > c_3 q^{-\kappa}$$

where $c_3 = c_3(d, \kappa, a, b)$. The constants $c_1, c_2, c_3$ are computable here. In another paper (1964b), Baker proved among other results that

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{10^{-6}}{q^{2.955}}.$$

These results of Baker improve the exponent in Liouville's Theorem for certain algebraic numbers. Using his estimates of linear forms whose coefficients are logarithms of algebraic numbers, Baker also proved a result which holds for *all* real algebraic numbers and which improves Liouville's Theorem by a factor which is smaller than any positive power of $q$:

THEOREM 5A (Baker 1968b). *Suppose $\alpha$ is a real algebraic number of degree $d \geqq 3$ and suppose $\kappa > d$. Then there is a computable $c_4 = c_4(\alpha, \kappa) > 0$ such that*

$$\left| \alpha - \frac{p}{q} \right| > c_4 e^{(\log q)^{1/\kappa}} q^{-d}$$

*for every rational $\frac{p}{q}$ with $q > 0$.*

Hence if $f(q)$ is of smaller order of magnitude than $e^{(\log q)^{1/\kappa}}$ for some $\kappa > d$, say if $f(q) \leqq e^{(\log q)^{1/(d+\delta)}}$ where $\delta > 0$, then the solutions $\frac{p}{q}$ of $\left| \alpha - \frac{p}{q} \right| < f(q) q^{-d}$ must have $q \leqq q_1 = q_1(\alpha, \delta)$ where $q_1$ is computable. Recently Baker and Stark (to appear) could replace $\kappa > d$ by the milder condition $\kappa > 1$.

Feldman (1968a, 1968b) proved a result which contains the following.

THEOREM 5B. *Suppose $\alpha$ is an irrational algebraic number and let $p_1, \ldots, p_r$ be distinct rational primes. Then for all rationals $\frac{p}{q}$ with $p \geqq 3$, $q \geqq 3$, $(p, q) = 1$, of the type*

$$\frac{p}{q} = p_1^{a_1} \ldots p_r^{a_r}$$

*with rational integers $a_1, \ldots, a_r$, one has*

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{(\log q)^{c_5}},$$

*where $c_5$ is an effectively computable constant depending on $\alpha, p_1, \ldots, p_r$.*

This sharpens the case $p' = q' = 1$ of Theorem 4F. To prove Theorem 5B, Feldman used the method of Baker and refined it in the special case needed here. Baker's (1966, 1967b) papers would have yielded $\left| \alpha - (p/q) \right| > c_6 e^{-(\log \log q)^{\kappa}}$.

**5.2.** As for Thue's equation, the following effective theorem holds.

THEOREM 5C (Baker 1968b, 1968c). *Suppose the form $F(x, y)$ in Thue's equation*

$$F(x, y) = m \tag{5.1}$$

*is of degree $d \geqq 3$, it has rational integer coefficients and is irreducible over the rationals. Then every integer solution $(x, y)$ of this equation satisfies*

$$\max(|x|, |y|) < \exp\left((dH)^{(10d)^5} + (\log m)^{2d+2}\right),$$

*where $H$ is the height of $F$.*

Baker also gave explicit bounds for the solutions of elliptic and hyperelliptic equations (1968c, 1968d, 1969) and Baker and Coates (1970) did the same for equations which define curves of genus 1. Vinogradov and Sprindžuk (1968), Coates (1969, 1970a, 1970b) and Sprindzuk (1970b) used Baker's method to prove effective $p$-adic theorems.

Sprindžuk (1969, 1970a) used a $p$-adic method for estimating the size of the integer solutions $x, y, z_1, \ldots, z_r$ of the equation

$$F(x, y) = m p_1^{z_1} \ldots p_r^{z_r} \qquad (x, y) = 1, z_1 \geqq 0, \ldots, z_r \geqq 0 \tag{5.2}$$

where $F(x, y)$ is an irreducible form of degree $d \geqq 3$ and where $p_1, \ldots, p_r$ are rational primes. In (1970b) he improved these results. He defined *exceptional* forms $F(x, y)$ and showed that if $F(x, y)$ is not exceptional then $\max(|x|, |y|) < c_1 \exp(\log |m|)^{\kappa}$ where $\kappa > 2$ and $c_1 = c_1(F, \kappa, p_1, \ldots, p_r)$ is an effective constant independent of $m$. He further improved his results in (1971a). He showed that there are no exceptional forms of degree $d \geqq 5$ and he proved the following:

THEOREM 5D. *Suppose $F(x, y)$ is not an exceptional form. Then all the integer solutions of* (5.2) *satisfy*

$$\max(|x|, |y|) < c_2 \mid m \mid^{(\log\log|m|)^{4(d+r+1)}}. \tag{5.3}$$

*Here $c_2 = c_2(F, p_1, \ldots, p_r)$ is effective.*

A full account of this work is given by Sprindžuk (to appear).

Baker (to appear) further improves this estimate for the more special equation $F(x, y) = m$ but for all irreducible forms $F$ of degree $d \geqq 3$ without exception, and derives the estimate

$$\max(|x|, |y|) < c_3 \mid m \mid^{\log\log|m|}.$$

It is almost certain that this estimate can be extended to the more general equation (5.2). On the other hand Sprindžuk at the end of his (1971a) paper indicates that his method can be used to replace (5.3) by the still sharper inequality

$$\max(|x|, |y|) < c_4 \mid m \mid^{c_5}.$$

THEOREM 5E (Sprindžuk 1971b). *Suppose the binary form $F(x, y)$ of degree $d \geqq 3$ is not exceptional. Let $x, y$ be coprime integers with $X = \max(|x|, |y|) > 10$. Then the greatest prime factor of $F(x, y)$ is*

$$> c_6 \log\log X / \log\log\log X \tag{5.4}$$

*where $c_6 = c_6(F)$ is effectively computable.*

Earlier Coates (1970a) had given the lower bound $c_7 (\log\log X)^{1/4}$ which holds for all irreducible forms of degree $d \geqq 3$. Probably it is possible to generalize Baker's paper (to appear) to the $p$-adic case, and then to prove the estimate of Theorem 5E for all irreducible forms of degree $d \geqq 3$. Mahler's Theorem 4E had said that the greatest prime factor of $F(x, y)$ tends to infinity as $X \to \infty$.

Now suppose $F(x)$ is a polynomial in one variable $x$ with rational integer coefficients, of degree $d \geqq 2$ and with distinct roots. Keates (1969) shows that the greatest prime factor $p(x)$ of $F(x)$ is $> c_8 \log\log|x|$ if $F(x)$ is of some special type, e.g. if $F(x)$ is of degree $d = 2$ or 3. Combining Keates' argument with recent papers of Baker and Sprindžuk it might be possible to show that $p(x) > c_8 \log\log|x|$ in general. It is interesting that this bound is only slightly better than the inequality (5.4) for forms in two variables. Further references on $p(x)$ are given in Keates (1969).

For an effective version of Theorem 4G see Sprindžuk (1970b, 1971a).

**5.3.** Baker derived Theorem 5A from his deep lower bounds for expressions of the type

$$|\beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n| \tag{5.5}$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, which he developed in (1966, 1967b, 1967c, 1968a).

Namely, suppose $\left|\alpha - \frac{p}{q}\right|$ is small and put $\beta = q\alpha - p$. Let $\mathbf{Q}(\alpha)$ be the field obtained by adjoining $\alpha$ to the field $\mathbf{Q}$ of rationals, let $\alpha^{(1)} = \alpha, \alpha^{(2)}, \ldots, \alpha^{(d)}$ be the conjugates of $\alpha$ and for $\omega \in \mathbf{Q}$ let $\omega^{(1)}, \ldots, \omega^{(d)}$ be the conjugates of $\omega$ corresponding to $\alpha^{(1)}, \ldots, \alpha^{(d)}$. We have

$$(\alpha^{(j)} - \alpha^{(k)})\beta^{(l)} + (\alpha^{(k)} - \alpha^{(l)})\beta^{(j)} + (\alpha^{(l)} - \alpha^{(j)})\beta^{(k)} = 0$$

for any integers $j, k, l$ with $1 \leqq j, k, l \leqq d$. Let $\gamma$ be an associate of $\beta$, of the type

$$\gamma = \beta \eta_1^{b1} \ldots \eta_r^{br}$$

where $\eta_1, \ldots, \eta_r$ is a fixed set of fundamental units of $\mathbf{Q}(\alpha)$ and where $b_1, \ldots, b_r$ are rational integers. We have

$$\frac{(\alpha^{(k)} - \alpha^{(l)})\beta^{(j)}}{(\alpha^{(j)} - \alpha^{(l)})\beta^{(k)}} - 1 = \frac{(\alpha^{(k)} - \alpha^{(j)})\beta^{(l)}}{(\alpha^{(j)} - \alpha^{(l)})\beta^{(k)}},$$

whence

$$\alpha_1^{b1} \ldots \alpha_r^{br} \alpha_{r+1}^{-1} - 1 = \sigma \tag{5.6}$$

where

$$\alpha_s = \eta_s^{(k)}/\eta_s^{(j)} \; (1 \leqq s \leqq r), \quad \alpha_{r+1} = \frac{(\alpha^{(j)} - \alpha^{(l)})\gamma^{(k)}}{(\alpha^{(k)} - \alpha^{(l)})\gamma^{(j)}}$$

and

$$\sigma = \frac{(\alpha^{(k)} - \alpha^{(j)})\,\beta^{(l)}}{(\alpha^{(j)} - \alpha^{(l)})\,\beta^{(k)}}.$$

Now $|\beta| = |\beta^{(1)}|$ is small by hypothesis, and it is clear that there is a conjugate $\beta^{(k)}$ with $|\beta^{(k)}| \geqq c_0(\alpha)|q|$. We now put $l = 1$ and pick $j$ distinct from $k, l$. The quotient $|\beta^{(l)}/\beta^{(k)}|$ and hence $|\sigma|$ is then small. Therefore the left hand side of (5.6) will be small and

$$|b_1 \log \alpha_1 + \ldots + b_r \log \alpha_r - \log \alpha_{r+1} - k\pi i|$$

will be small for some integer $k$. Since $\pi i = \log(-1)$, this expression is of the type (5.5). One can choose the associate $\gamma$ of $\beta$ such that all the quotients $|\gamma^{(k)}/\gamma^{(j)}|$ $(1 \leqq k, j \leqq d)$ are bounded independently of $p$, $q$, and hence $\alpha_{r+1}$ as well as $\alpha_1, \ldots, \alpha_r$ and their conjugates are bounded. Substituting explicit values for the estimates and using his lower bounds for (5.5), Baker obtains a contradiction if $\beta = q\alpha - p$ is too small, and thereby he proves Theorem 5A.

A more quantitative discussion of this argument as it applies in the proof of Theorem 5C is given by Baker (1971). There is an anticipation of the argument at the end of Gelfond's (1952) book. Gelfond dealt with certain cubic Thue equations $F(x, y) = 1$ and pointed out that a lower bound for (5.5) (which then was not known) would provide upper bounds for the size of solutions of these equations.

## 6. Simultaneous approximation to real numbers by rationals

**6.1.** In this section we shall provide the background for the more special problem of simultaneous approximation to real algebraic numbers, which will be discussed in §7. Using the same general principles that were used in the proof of Theorem 1A and its corollary, Dirichlet (1842) proved the following two theorems and their corollaries.

Theorem 6A. *Let $\alpha_1, \ldots, \alpha_l$ be real numbers and suppose $Q$ is an integer greater than* 1. *Then there exist integers $q, p_1, \ldots, p_l$ with*

$$(6.1) \qquad 1 \leqq q < Q^l \quad \text{and} \quad |\alpha_i q - p_i| \leqq Q^{-1} \qquad (i = 1, \ldots, l).$$