11. Generalizations and open problems

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discuss what happens in the case (B). For simplicity we shall assume that K is totally real.

There is an element φ of the Galois group such that the sets $\{\alpha^{(1)}, \ldots, \alpha^{(\mu)}\}$ and $\{\varphi(\alpha^{(1)}), \ldots, \varphi(\alpha^{(\mu)})\}$ are neither identical nor disjoint. We may assume without loss of generality that

$$\{\varphi(\alpha^{(1)}), \ldots, \varphi(\alpha^{(\mu)})\} = \{\alpha^{(1)}, \ldots, \alpha^{(l)}, \alpha^{(\mu+1)}, \ldots, \alpha^{(2\mu-l)}\}.$$

Here $1 \leqq l \leqq \mu - 1$. The forms $L^{(1)}, \ldots, L^{(\mu)}$ have rank ρ, and hence also $L^{(1)}, \ldots, L^{(l)}, L^{(\mu+1)}, \ldots, L^{(2\mu-l)}$ have rank ρ. Denote the rank of $L^{(1)}, \ldots, L^{(l)}$ by r_1 and the rank of $L^{(1)}, \ldots, L^{(\mu)}, \ldots, L^{(2\mu-l)}$ by r_2. It is easily seen that $r_2 \leqq 2\rho - r_1$, i.e. that

$$r_1 + r_2 \leqq 2\rho .$$

Since μ was chosen as small as possible with (10.10), and since $l \leqq \mu - 1$, we have $l < r_1 q$. The number $2\mu - l$ of elements of $L^{(1)}, \ldots, L^{(\mu)}, \ldots, L^{(2\mu-l)}$ satisfies $2\mu - l \leqq r_2 q$ by (10.9). Thus

$$2\mu = l + (2\mu - l) < r_1 q + r_2 q \leqq 2\rho q ,$$

which contradicts (10.10). Hence (B) is impossible if K is totally real.

We have in fact used the hypothesis that K is totally real, for in general $L^{(1)}, \ldots, L^{(l)}$ need not be a Symmetric System, and $l < r_1 q$ need not hold. The situation is therefore somewhat more complicated if K is not totally real.

11. Generalizations and open problems

11.1. The theorems of §7 and §10 can almost certainly be generalized to include p-adic valuations. I understand that work on this question is being done now. (p-adic versions of the results of §2 were discussed in §4.5). Next, suppose that K is an algebraic number field and that $\alpha_1, \ldots, \alpha_l$ are algebraic numbers such that $1, \alpha_1, \ldots, \alpha_l$ are linearly independent over K. It is likely that *for every $\delta > 0$ there are only finitely many l-tuples of elements $\beta_1, \ldots, \beta_l$ of K with*

$$(11.1) \qquad |\alpha_i - \beta_i| < \mathscr{H}(\beta)^{-1-(1/l)-\delta} \quad (i = 1, \ldots, l),$$

where $\mathscr{H}(\beta)$ is a suitably defined height of $\beta = (\beta_1, \ldots, \beta_l)$. A possible definition for $\mathscr{H}(\beta)$ is

$$\mathscr{H}(\beta) = \prod_v \max(1, \|\beta_1\|_v, \ldots, \|\beta_l\|_v),$$

where v runs through the valuations of K and where $\| \|_v$ is defined as in §4.5. In view of (4.9), $\mathscr{H}(\beta)$ is almost the same as $H_K(\beta)$ if $l = 1$, and hence a theorem on (11.1) would generalize Le Veque's Theorem 4A. One could try to obtain a still more general theorem which would contain both the p-adic case and the case of a number field K. Such a result was put forward as a conjecture by Lang (1962, Ch. 6).

11.2. We have already said in §2.2 that it would be desirable to replace the factor q^δ in Roth's Theorem by something smaller, say by a power of $\log q$. The same is true of the generalizations of Roth's Theorem to simultaneous approximation.

The theorems of §2, 7 and 10 are non-effective. For approximation to a single algebraic number α there are the effective results of Baker (see §5), but for simultaneous approximation there are only the relatively special effective theorems of Baker (1967a), Feldman (1970a, 1970b) and Osgood (1970).

11.3. The following questions also appear to be very difficult. Suppose $(\alpha_1, ..., \alpha_l)$ is a point of transcendence degree $d < l$. The theorems of §7 deal with the case when the point is algebraic, i.e. when $d = 0$. What can one say for other values of d? Perron (1932) and Schmidt (1962) obtained results, of about the same level of sophistication as Liouville's Theorem, which can be used to show that certain given points have transcendence degree l.

A better question perhaps is how close rational points can come to a given algebraic variety. We may reformulate this question in a homogeneous setting. Let V be a homogeneous variety defined over the rationals (i.e. one defined by homogeneous polynomial equations with rational coefficients) in E^n with $n \geqq 2$. For every $\mathbf{x} \neq \mathbf{0}$ we put

$$\psi(V, \mathbf{x}) = \Delta(V, \mathbf{x}) \, |\mathbf{x}|^{-1}$$

where $\Delta(V, \mathbf{x})$ is the distance from $\mathbf{x}$ to V. It is clear that $\psi(V, \lambda\mathbf{x}) = \psi(V, \mathbf{x})$; the function $\psi(V, \mathbf{x})$ may be interpreted as the "angle" between V and the vector $\mathbf{x}$. We are interested in inequalities of the type

$$\psi(V, \mathbf{x}) < c \, |\mathbf{x}|^{-\omega} \tag{11.2}$$

where $\mathbf{x}$ runs through the integer points. We saw in §6 that Theorems 6A and 6C had such an interpretation. The best value of ω for which (11.2) has infinitely many integer solutions can always be found if V is linear, i.e. is a subspace. For the non-linear case we have neither a good generalization of Dirichlet's Theorem nor anything like Roth's Theorem.

Suppose now that V is a hypersurface containing no integer point $\mathbf{x} \neq \mathbf{0}$ and defined by the equation $F(\mathbf{x}) = 0$ where F is a form of degree d with rational integer coefficients. For every integer point $\mathbf{x} \neq \mathbf{0}$ we have $|F(\mathbf{x})| \geqq 1$, and since $\left|\frac{\partial}{\partial x_i} F(\mathbf{x})\right| \leqq c_1 |\mathbf{x}|^{d-1}$ $(i=1, \ldots, n)$, the distance from $\mathbf{x}$ to V is $\geqq c_2 |\mathbf{x}|^{1-d}$, which in turn implies that

$$\psi(V, \mathbf{x}) \geqq c_3 |\mathbf{x}|^{-d},$$

where the constants depend only on V. This inequality may be interpreted as a generalization of Liouville's Theorem. Any improvement of this inequality, even though perhaps it may apply only to special classes of non-linear hypersurfaces, would be of great interest and would shed light on certain diophantine equations different from the equations with norm forms discussed in §10.

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