§2. The cusps and their resolution for the 2-dimensional case

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Problem. Prove these congruences in the framework of elementary number theory.

d	2	3	4	5	6	$2\zeta_K(-1)$	2	3	4	6	12	$e(\mathfrak{H}^2/G)$	$e(\mathfrak{H}^2/\hat{G})$
2	2	2	2			1/6	—	—	—	—	—	4	$G = \hat{G}$
3	3	2			1	1/3	3	1	1		1	4	4
5	2	2		2		1/15	—	—	—	—	—	4	$G = \hat{G}$
6	6	3				1	5	1	2	1		6	6
7	4	4				4/3	5	2	2			6	6
10	6	4				7/3	—	—	—	—	—	8	$G = \hat{G}$
11	10	4				7/3	5	2	4			10	8
13	2	4				1/3	—	—	—	—	—	4	$G = \hat{G}$
14	12	4				10/3	8	2	4			12	10
15	8	6				4	—	—	—	—	—	12	?
17	4	2				2/3	—	—	—	—	—	4	$G = \hat{G}$
19	10	4				19/3	9	2	4			14	12
21	4	5				2/3	3	2		1		6	4
22	6	8				23/3	12	4	2			16	14
23	12	8				20/3	7	4	6			18	14
26	18	4				25/3	—	—	—	—	—	20	$G = \hat{G}$
29	6	6				1	—	—	—	—	—	8	$G = \hat{G}$
30	12	10				34/3	—	—	—	—	—	24	?
31	12	4				40/3	11	2	6			22	18
33	4	3				2	7	1		1		6	6
34	12	4				46/3	—	—	—	—	—	24	?
35	20	8				38/3	—	—	—	—	—	28	?
37	2	8				5/3	—	—	—	—	—	8	$G = \hat{G}$
38	18	8				41/3	16	4	6			28	22
39	16	10				52/3	—	—	—	—	—	40	?
41	8	2				8/3	—	—	—	—	—	8	$G = \hat{G}$

§ 2. THE CUSPS AND THEIR RESOLUTION FOR THE 2-DIMENSIONAL CASE

2.1. Let K be a totally real algebraic field of degree n over $\mathbf{Q}$ and M an additive subgroup of K which is a free abelian group of rank n. Such a group M is called a complete $\mathbf{Z}$-module of K. Let U_M^+ be the group of those units ε of K which are totally positive and satisfy $\varepsilon M = M$. Any $\alpha \in K$ with $\alpha M = M$ is automatically an algebraic integer and a unit.

The group U_M^+ is free of rank $n - 1$ (compare [6]).

Two modules M_1, M_2 are called (strictly) equivalent if there exists a (totally positive) number $\lambda \in K$ with $\lambda M_1 = M_2$. Of course, $U_{M_1}^+ = U_{M_2}^+$ for equivalent modules.

According to [71] p. 45, Theorem 4, for any parabolic point x of an irreducible discrete subgroup Γ of $(\mathbf{PL}^+(\mathbf{R}))^n$ with $\mathfrak{H}^n/\Gamma$ of finite volume the element $\rho \in (\mathbf{PL}^+(\mathbf{R}))^n$ with $\rho x = \infty$ can be chosen in such a way that the group $\rho\Gamma_x\rho^{-1}$ (see 1.5 (15)) is contained in $\mathbf{PL}^+(K) \subset (\mathbf{PL}^+(\mathbf{R}))^n$ where K is a suitable totally real field. Then we have an exact sequence

$$0 \to M \to \rho\,\Gamma_x\rho^{-1} \to V \to 1$$

where M is a complete $\mathbf{Z}$-module in K and V is a subgroup of U_M^+ of rank $n-1$. The field K, the strict equivalence class of M and the group V are completely determined by the parabolic orbit and do not depend on the choice of ρ.

It can be shown more generally ([71] p. 45, footnote 3) that there exists a $\rho \in (\mathbf{PL}_2^+(\mathbf{R}))^n$ such that $\rho\Gamma\rho^{-1} \subset \mathbf{PL}_2^+(K)$, provided there is at least one parabolic orbit. Therefore, the field K is the same for all parabolic orbits. The conjecture of Selberg (1.5 Remark) remains unsettled, because, if we represent the elements of $\rho\Gamma\rho^{-1}$ by matrices with coefficients in $\mathfrak{o}_K$, we have no information on the determinants of these matrices.

A parabolic orbit will be called a *cusp*. We say that the cusp is of *type* (M, V). If x is a point in the parabolic orbit, we often say that the cusp is at x. Sometimes the cusp will be denoted by x.

For a given pair (M, V) with $V \subset U_M^+$ (where V has rank $n-1$) we define

$$G(M, V) = \left\{ \left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & 1 \end{smallmatrix}\right) \,\middle|\, \varepsilon \in V, \mu \in M \right\} = M \rtimes V \quad \text{(semi-direct product)}$$

For $n = 2$, the element $\rho \in \mathbf{PL}_2^+(\mathbf{R})^n$ can be chosen in a such a way that $\rho\Gamma_x\rho^{-1} = G(M, V)$.

Let K be a totally real field of degree n over $\mathbf{Q}$, let M be a complete $\mathbf{Z}$-module in K and V a subgroup of U_M^+ of finite index. Suppose $\mathfrak{G}$ is a group of matrices $\left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & 1 \end{smallmatrix}\right)$ (with $\varepsilon \in V$, $\mu \in K$, and $\mu \in M$ for $\varepsilon = 1$) such that the sequence

$$0 \to M \to \mathfrak{G} \to V \to 1 \tag{1}$$

is exact.

The group $\mathfrak{G}$ operates freely and properly discontinuously on $\mathfrak{H}^n$. We add one additional point ∞ to the complex manifold $\mathfrak{H}^n/\mathfrak{G}$. A complete system of open neighborhoods of ∞ in the new space $\overline{\mathfrak{H}^n/\mathfrak{G}} = \mathfrak{H}^n/\mathfrak{G} \cup \infty$ is given by the sets

$$(2) \qquad (\overset{\circ}{W}(d) / \mathfrak{G}) \cup \infty$$

where, for any positive d,

$$(3) \qquad \overset{\circ}{W}(d) = \{ z \mid z \in \mathfrak{H}^n, \prod_{j=1}^{n} \operatorname{Im}(z_j) > d \}$$

The local ring $\mathfrak{O}(\mathfrak{G})$ at ∞ is defined as the ring of functions holomorphic in some neighborhood of ∞ (except ∞) and continuous in ∞. For $n > 1$ the condition "continuous in ∞" can be dropped ([71], p. 50, lemma 7).

If $\mathfrak{G} = G(M, V)$ we put $\mathfrak{O}(\mathfrak{G}) = \mathfrak{O}(M, V)$. We shall only give the structure of $\mathfrak{O}(M, V)$ explicitly. *For $n = 2$ this is no loss of generality.* The ring $\mathfrak{O}(M, V)$ has the following structure:

Let M^* be the complete module in K which is dual to M: An element $x \in K$ belongs to M^* if and only if the trace $\operatorname{tr}(xa)$ is an integer for all $a \in M$. We recall that

$$\operatorname{tr}(xa) = \sum_{j=1}^{n} x^{(j)} a^{(j)}$$

Let M^{*+} be the set of all totally positive elements of M. The local ring $\mathfrak{O}(M, V)$ is the ring of all Fourier series

$$(4) \qquad f = a_0 + \sum_{x \in M^{*+}} a_x \cdot e^{2\pi i (x^{(1)} z_1 + \ldots + x^{(n)} z_n)},$$

for which the coefficients a_x satisfy $a_{\varepsilon x} = a_x$ for all $\varepsilon \in V$, and which converge on $\overset{\circ}{W}(d)$ for some positive d depending on f.

Proposition. The space $\overline{\mathfrak{H}^n/\mathfrak{G}}$ with the local ring $\mathfrak{O}(\mathfrak{G})$ at ∞ is a normal complex space.

This is known for $n = 1$, of course. For $n \geqq 2$ we have to check H. Cartan's condition ([67] Exposé 11, Théorème 1) that there is some neighbourhood U of ∞ such that for any two different points $p_1, p_2 \in U - \{\infty\}$ there exists a holomorphic function f in $U - \{\infty\}$ with $f(p_1) \neq f(p_2)$. If $\mathfrak{G}$ occurs as group $\rho \Gamma_x \rho^{-1}$ for some cusp of a group Γ satisfying condition (F) of 1.5, Cartan's condition is proved in the theory of compactification (0.3) by the use of Γ-automorphic forms. The group $G(M, U_M^+)$ occurs in such a way. Namely, M is strictly equivalent to an ideal in some order $\mathfrak{o}$ of K (see [6]) where $\mathfrak{o} = \{x \in K \mid x M \subset M\}$. Therefore, we may assume that M is such an ideal. The cusp at ∞ of the arithmetic group (commensurable with the Hilbert modular group)

$$\{ \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \mid \alpha, \beta, \gamma, \delta \in \mathfrak{o}, \beta \in M, \alpha\delta - \beta\gamma \in U_M^+ \}$$

has the isotropy group $G(M, U_M^+)$.

As W. Meyer pointed out to me, the group $H^2(V, M)$ —the set of all equivalence classes of extensions over V with kernel M and belonging to the action of V on M—is finite. (It vanishes for $n \leqq 2$.) This implies the existence of a translation $\rho \in \mathbf{PL}_2^+(K)$ with $\rho z = z + a$ such that $\rho \mathfrak{G} \rho^{-1} \subset G(\tilde{M}, V)$ where $\tilde{M} = \frac{1}{k} M$ and k is the order of the extension $\mathfrak{G}$ as element of $H^2(V, M)$. Therefore $\rho \mathfrak{G} \rho^{-1}$ is commensurable with $G(M, U_M^+)$, and it follows from general results on ramifications of complex spaces [18] that $\mathfrak{H}^n/\mathfrak{G}$ is a normal complex space. (See also 0.7 for quotients of normal complex spaces).

Remark. It would be interesting to check Cartan's condition directly using only the structure of the ring $\mathfrak{O}(\mathfrak{G})$. It seems to be unknown if every $\mathfrak{G}$ occurs for a cusp of a group Γ of type (F). We shall call the point ∞ of the normal complex space $\overline{\mathfrak{H}^n/\mathfrak{G}}$ a "cusp", even if it does not occur for a group Γ.

The point ∞ (with the local ring $\mathfrak{O}(\mathfrak{G})$) is non-singular for $n = 1$. Probably it is always singular for $n \geqq 2$. This was shown by Christian [11] to be true for the cusps of the Hilbert modular group of a totally real field of degree $n \geqq 2$. For $n = 2$, see [21].

Our aim is to resolve the point ∞ of $\overline{\mathfrak{H}^2/G(M, V)}$ in the sense of the theory of resolution of singularities in a normal complex space of dimension 2 (see, for example, [35], [49]). This will be done in 2.4 and 2.5. The resolution process shows that ∞ is always a singular point.

It remains an open problem to give explicit resolutions also for $n > 2$.

If Γ is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ satisfying the condition (F) of the definition in 1.5, then $\mathfrak{H}^n/\Gamma$ can be compactified by adding t points (cusps) where t is the number of Γ-inequivalent parabolic points of Γ. The resulting space is a compact normal complex space. It is even a projective algebraic variety (0.3).

2.2. In the next sections we shall consider the case $n = 2$, construct certain normal singularities of complex surfaces and show that they are cusps in the sense of 2.1. The construction will be very much related to continued fractions.

Consider a function $k \mapsto b_k$ from the integers to the natural numbers greater or equal 2. For each integer k take a copy R_k of $\mathbf{C}^2$ with coordinates u_k, v_k. We define R_k' to be the complement of the line $u_k = 0$ and R_k'' to be the complement of $v_k = 0$. The equations

$$(5) \qquad \begin{aligned} u_{k+1} &= u_k^{\,b_k} v_k \\ v_{k+1} &= {}^1/u_k \end{aligned}$$

give a biholomorphic map $\varphi_k : R_k' \to R_{k+1}''$

In the disjoint union $\cup R_k$ we make all the identifications (5). We get a set Y. We may now consider each R_k as a subset of Y. Each R_j is mapped by (u_j, v_j) bijectively onto $\mathbf{C}^2$. This defines an atlas of Y. A subset of Y is open if and only if its intersection with each R_j is an open subset of R_j.

Lemma. The topological space Y defined by (5) *satisfies the Hausdorff separation axiom.*

Proof. Denote the map $R_j \to \mathbf{C}^2$ by ψ_j. Let k be an integer. According to Bourbaki [7] p. 36, we have to show that the graph of

$$(6) \qquad \psi_{j+k} \circ \psi_j^{\,-1} : \psi_j(R_j \cap R_{j+k}) \to \psi_{j+k}(R_j \cap R_{j+k})$$

is closed in $\psi_j(R_j) \times \psi_{j+k}(R_{j+k}) = \mathbf{C}^2 \times \mathbf{C}^2$. Without loss of generality we may assume $j = 0$ and $k > 0$. The map $\psi_k \circ \psi_0^{\,-1}$ is given by

$$(7) \qquad \begin{aligned} u_k &= u_0^{\,p_k} \cdot v_0^{\,q_k} \\ v_k &= u_0^{\,-p_{k-1}} \cdot v_0^{\,-q_{k-1}} \end{aligned}$$

where

$$\begin{pmatrix} p_k & q_k \\ -p_{k-1} & -q_{k-1} \end{pmatrix} = \begin{pmatrix} b_{k-1} & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b_{k-2} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\frac{p_k}{q_k} = b_0 - \frac{1}{b_1} - \cdots - \frac{1}{b_{k-1}}$$

p_k, q_k are coprime. We define $p_0 = 1$, $q_0 = 0$ and have

$$p_{k+1} = b_k p_k - p_{k-1} \qquad \text{for } k \geqq 1\,,$$
$$q_{k+1} = b_k p_k - p_{k+1} \qquad \text{for } k \geqq 1\,,$$
$$p_k > q_k, p_{k+1} > p_k \geqq 1, q_{k+1} > q_k \geqq 0, \quad \text{for } k \geqq 0\,.$$

The intersection $R_0 \cap R_k$ as subset of R_0 is given by $u_0 \neq 0$, $v_0 \neq 0$ for $k \geqq 2$ and by $u_0 \neq 0$ for $k = 1$. The graph of $\psi_k \cdot \psi_0^{-1}$ (see (6)) is given by

$$\begin{aligned} &u_k = u_0^{p_k} \cdot v_0^{q_k},\ v_k \cdot u_0^{p_{k-1}} \cdot v_0^{q_{k-1}} = 1 \\ &u_0 \neq 0, \quad v_0 \neq 0 \quad (k \geqq 2) \\ &u_0 \neq 0 \qquad\qquad\quad (k = 1) \end{aligned}$$

But the inequalities follow from the equations. Therefore the graph is closed in $\mathbf{C}^2 \times \mathbf{C}^2$. This finishes the proof of the lemma. The negative exponents in the second line of (7) were essential.

The argument would break down, for example, if $k = 6$ and $b_i = 1$ for $0 \leqq i \leqq 5$, because $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)^6 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

The topological space Y obviously has a countable basis. For any function $k \mapsto b_k \geqq 2$ we have constructed a complex manifold Y of complex dimension 2. In Y we have a string of compact rational curves S_k non-singularly embedded ($k \in \mathbf{Z}$). The curve S_k is given by $u_{k+1} = 0$ in the $(k+1)$-th coordinate system and by $v_k = 0$ in the k-th coordinate system. S_k, S_{k+1} intersect in just one point transversally, namely in the origin of the $(k+1)$-th coordinate system. S_i, S_k $(i<k)$ do not intersect, if $k - i \neq 1$. The union of all the S_k is a closed subset of Y.

Lemma. The self-intersection number of the curve S_k equals $- b_k$.

Proof. The coordinate function u_{k+1} extends to a meromorphic function on Y. Its divisor is an infinite integral linear combination of the S_j which because of (5) contains S_{k-1} with multiplicity b_k, the curve S_k with multiplicity 1 and the curve S_{k+1} with multiplicity 0. The intersection number of S_k with this divisor is zero. Since it is also equal to $b_k + S_k \cdot S_k$, the result follows.

Remark. The construction of Y is analogous to the resolution of a quotient singularity in [35], 3.4. For technical reasons we have changed the notation by shifting the indices of S_k and b_k by 1. This should also be taken into account when comparing with [39], § 4.

2.3. Let us assume that the function $k \mapsto b_k \geqq 2$ of 2.2 is periodic, i.e. there exists a natural number $r \geqq 1$ such that

$$b_{k+r} = b_k.$$

Continued fractions of the form

$$a_0 - \cfrac{1}{a - \ddots - \cfrac{1}{a_s}}$$

shall be denoted by $[[a_0, ..., a_s]]$; similarly, $[[a_0, a_1, a_2, ...]]$ stands for infinite continued fractions of this kind. For our given function $k \mapsto b_k \geqq 2$ we consider the numbers

$$w_k = [[b_k, b_{k+1}, \ldots]], \quad k \in \mathbf{Z}. \tag{8}$$

The w_k are all equal to 1 if $b_j = 2$ for all j. *Therefore, we assume* $b_j \geqq 3$ *for at least one* j. Then all w_k are quadratic irrationalities which are greater than 1. They satisfy $w_{k+r} = w_k$ and all belong to the same real quadratic field K. We consider the complete $\mathbf{Z}$-module

$$M = \mathbf{Z}w_0 + \mathbf{Z} . 1 \subset K$$

Let $x \mapsto x'$ be the non-trivial automorphism of K. Thus $x = x^{(1)}$ and $x' = x^{(2)}$ in the notation of 1.3. The module M acts freely on $\mathbf{C}^2$ by $(z_1, z_2) \mapsto (z_1 + a, z_2 + a')$ for $a \in M$. For our function $j \mapsto b_j \geqq 2$ we have constructed in 2.2 a complex manifold Y. We now define a biholomorphic map

$$\Phi : Y - \bigcup_{j \in z} S_j \to \mathbf{C}^2/M$$
$$\Phi : (u_0, v_0) \mapsto (z_1, z_2)$$

by

$$\begin{aligned} 2\pi i z_1 &= w_0 \log u_0 + \log v_0 \\ 2\pi i z_2 &= w_0' \log u_0 + \log v_0 \end{aligned} \tag{9}$$

The logarithms are defined modulo integral multiples of $2\pi i$, thus (z_1, z_2) is well-defined modulo M. Observe that

$$Y - \bigcup_{j \in z} S_j = \{ (u_0, v_0) \mid u_0 \neq 0, v_0 \neq 0 \} .$$

Since the determinant $\begin{vmatrix} w_0 & w_0' \\ 1 & 1 \end{vmatrix} \neq 0$, we can solve (9) for $\log u_0$ and $\log v_0$ and obviously have a biholomorphic map. The map Φ can be written down with respect to the k-th coordinate system ($k \in \mathbf{Z}$). The result is as follows.

Put $A_0 = 1$ and $A_{k+1} = w_{k+1}^{-1} \cdot A_k$. This defines A_k inductively for any *integer* k:

$$A_k = (w_1 w_2 \ldots w_k)^{-1} \text{ for } k \geqq 1, \; A_{-k} = w_0 w_{-1} \ldots w_{-k+1} \text{ for } k \geqq 1,$$
$$0 < A_{k+1} < A_k \text{ for } k \in \mathbf{Z}, \; A_k \neq 1 \text{ for } k \neq 0.$$

Formula (8) implies $w_k = b_k - \dfrac{1}{w_{k+1}}$ and

$$b_k A_k = A_{k-1} + A_{k+1} \tag{10}$$

For any integer k, the numbers A_{k-1}, A_k are a basis for M. From the coordinate transformations (5) we get the expression for the map Φ in the k-th coordinate system

$$\begin{aligned} 2\pi i z_1 &= A_{k-1} \cdot \log u_k + A_k \cdot \log v_k \\ 2\pi i z_2 &= A_{k-1}' \cdot \log u_k + A_k' \cdot \log v_k \end{aligned} \tag{11}$$

We had assumed that the b_j are periodic with period r which implies $w_{k+r} = w_k$ for any k. Therefore, A_r^{-1} equals the product of any r consecutive w_j which gives

$$\begin{aligned} A_{k+r} &= A_r A_k && \text{for any } k \in \mathbf{Z} \\ (A_r)^n &= A_{nr} && \text{for any } n \in \mathbf{Z}. \end{aligned} \tag{12}$$

This implies that $A_r M = M$. *Therefore* A_r *is an algebraic integer and a unit* $\neq 1$.

If we apply the non-trivial automorphism of K to the equation $w_k = b_k - \dfrac{1}{w_{k+1}}$ and use the periodicity we get

$$\begin{aligned} w_{k+1}'^{-1} &= b_k - \frac{1}{w_k'^{-1}} \\ w_{k+1}'^{-1} &= [[b_k, b_{k-1}, \ldots]] > 1 \end{aligned} \tag{13}$$

Therefore,

$$(14) \qquad 0 < w_k' < 1 < w_k \qquad \text{for } k \in \mathbf{Z}.$$

Thus the w_k and the A_k are totally positive. Let V be the (infinite cyclic) subgroup of U_M^+ generated by A_r. Thus we have associated to our function $j \mapsto b_j \geqq 2$ (at least one $b_j \geqq 3$) and the given period r (which need not be the smallest one) a pair (M, V) and a group $G(M, V)$ (see 2.1) which determines a cusp singularity. We shall use the complex manifold Y constructed in 2.2 for a resolution of this cusp singularity.

We restrict Φ to the open subset $\Phi^{-1}(\mathfrak{H}^2/M)$ of Y. According to (11) this set is given by

$$A_{k-1} \cdot \log|u_k| + A_k \cdot \log|v_k| < 0$$
$$A_{k-1}' \cdot \log|u_k| + A_k' \cdot \log|v_k| < 0$$

Since $v_k = 0$ or $u_{k+1} = 0$ for a point on S_k and the above inequalities do not depend on the coordinate system, it follows that

$$Y^+ = \Phi^{-1}(\mathfrak{H}^2/M) \cup \bigcup_{k \in \mathbf{Z}} S_k$$

is an open subset of Y. The group

$$V = \{(A_r)^n \mid n \in \mathbf{Z}\}$$

acts on Y^+ as follows:

$(A_r)^n$ sends a point with coordinates u_k, v_k in the k-th coordinate system to the point with the same coordinates in the $(k+nr)$-th coordinate system. Because of the periodicity $b_{j+r} = b_j$, this is compatible with the identifications (5). Therefore the action of the infinite cyclic group V on the complex manifold Y^+ is well-defined. We have the exact sequence

$$0 \to M \to G(M, V) \to V \to 1$$

Thus V acts on $\mathfrak{H}^2/M$. On the other hand we have a biholomorphic map

$$\Phi : Y^+ - \bigcup_{k \in \mathbf{Z}} S_k \to \mathfrak{H}^2/M$$

Lemma. The actions of V on Y^+ and $\mathfrak{H}^2/M$ are compatible with Φ.

Proof. If a point p has coordinates u_k, v_k in the k-th system, its image point (z_1, z_2) under Φ is given by (11). If we let A_r act on p, its image point is mapped under Φ (use formula (11) for the $(k+r)$-th coordinate system and (12)) to $(A_r z_1, A_r' z_2)$.

Lemma. The action of V on Y^+ is free and properly discontinuous.

Proof. In view of the preceding lemma the action is free on $Y^+ - \bigcup_{k \in \mathbf{Z}} S_k$. By A_r^n (assume $n \neq 0$) a point on S_k is mapped to a point on S_{k+nr}. If it is fixed, it will be an intersection point $S_{j-1} \cap S_j$ of two consecutive curves, but this point is carried to $S_{j+nr-1} \cap S_{j+nr}$.

To prove that V is properly discontinuous we must show that for points p, q on Y^+ there exist neighborhoods U_1 and U_2 of p and q such that $gU_1 \cap U_2 \neq \emptyset$ only for finitely many $g \in V$. Since V acts properly discontinuously on $\mathfrak{H}^2/M$ and $\bigcup_{k \in \mathbf{Z}} S_k$ is closed in Y^+, this is clear if p and q both do not belong to $\bigcup_{k \in \mathbf{Z}} S_k$. If $p \in \bigcup_{k \in \mathbf{Z}} S_k$ and $q \notin \bigcup_{k \in \mathbf{Z}} S_k$ we use the function Φ.

For $(z_1, z_2) \in \mathfrak{H}^2$ put $\rho(z_1, z_2) = \operatorname{Im} z_1 \cdot \operatorname{Im} z_2$ and set

$$U_1 = \{u \mid u \in Y^+, \rho\,\Phi(u) < \rho\,\Phi(p) + 1\},$$

and let U_2 be the complement of $\overline{U}_1$ in Y^+.

Then $U_1 \cap U_2 = \emptyset$ and $gU_1 = U_1$ for $g \in V$.

Now suppose both points p and q lie on $\bigcup_{k \in \mathbf{Z}} S_k$. It is sufficient to prove the existence of neighborhoods U_1 and U_2 of p and q such that $gU_1 \cap U_2 \neq \emptyset$ for only finitely many $g = (A_r)^n$ with $n < 0$. Recall that A_r generates V. If q lies on S_j and in the j-th coordinate system and p on S_k and in the k-th system, then a neighborhood U_2 of q is given by

$$0 \leqq |u_j| < \frac{1}{\varepsilon}, |v_j| < \varepsilon \quad \text{(for } \varepsilon \text{ sufficiently small).}$$

A neighborhood U_1 of p is given by

$$0 \leqq |u_k| < \frac{1}{\varepsilon}, |v_k| < \varepsilon \quad \text{(for } \varepsilon \text{ sufficiently small).}$$

Suppose that $|n| \geqq k - j + 1$. Then a point (u_k, v_k) in the k-th system is mapped under $(A_r)^n$ $(n<0)$ to a point (u_j, v_j) in the j-th system if and only if

$$(15) \qquad \begin{aligned} u_j &= u_k{}^a \cdot v_k{}^b \\ u_j \cdot u_k{}^c \cdot v_k^d &= 1 \end{aligned}$$

where a, b, c, d are non-negative integers and $c > d$. In fact $\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ is a matrix of type (7) depending on n, of course. If the points (u_j, v_j) and (u_k, v_k) lie in the chosen neighborhoods of p and q we obtain from (15) the inequality

$$\varepsilon^{d-c-1} < 1$$

which is not true for $\varepsilon \leqq 1$. Therefore, the image of U_1 under $(A_r)^n$ does not intersect U_2 for $n < 0$ and $|n| \geqq k - j + 1$.

Remark. The elements of $M = \mathbf{Z}w_0 + \mathbf{Z}$ can be written in the form $y - xw_0$ with $x, y \in \mathbf{Z}$. The number $y - xw_0$ is totally positive if and only if

$$y - xw_0 > 0 \quad \text{and} \quad y - xw_0' > 0$$

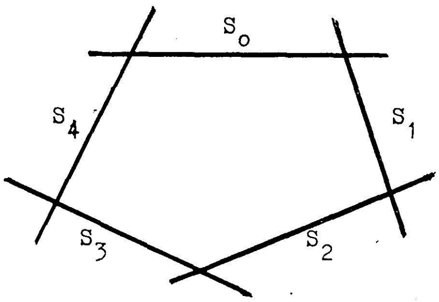
Since $w_0 > 1 > w_0' > 0$, the totally positive elements of M correspond exactly to the integral points in the (x, y)-plane which lie in the quadrant (angle $< 180°$) bounded by $y - xw_0 = 0$ $(x \geqq 0)$ and $y - xw_0' = 0$ $(x \leqq 0)$. If we write $A_k = p_k - q_k w_0$, then for $k \geqq 0$ these are the p_k, q_k of 2.2. We have

$$\lim_{k \to \infty} \frac{p_k}{q_k} = w_0, \qquad \lim_{k \to -\infty} \frac{p_k}{q_k} = w_0'$$

More precisely, it can be shown [12] that the A_k are exactly the lattice points of the support polygon, i.e. the polygon which bounds the convex hull of the lattice points in the above quadrant. It follows [12] that every totally positive number of M can be written uniquely as a linear combination of one or of two consecutive numbers A_k with positive integers as coefficients.

2.4. In section 2.3 we have constructed for a periodic function $k \mapsto b_k \geqq 2$ (with $b_j \geqq 3$ for at least one j) a complex manifold Y^+ together with a free properly discontinuous action of an infinite cyclic group V on Y^+. The orbit space Y^+/V is a complex manifold. The curve S_k in Y^+ was mapped by the generator A_r of V onto the curve S_{k+r} where r was the period. Thus S_k and S_{k+r} become the same curve in Y^+/V. We shall denote the curves in Y^+/V again by S_k $(k \in \mathbf{Z})$ with the understanding that we have $S_k = S_{k+r}$. We have in Y^+/V for $r \geqq 3$ a cycle $S_0, S_1, \ldots, S_{r-1}$ of non-singular rational curves such that S_k and S_{k+1} intersect transversally in exactly one point ($k \in \mathbf{Z}/r\mathbf{Z}$) and the selfintersection number $S_k \cdot S_k$ equals $-b_k$. Otherwise there are no intersections. The configuration is illustrated by the diagram:

(16)

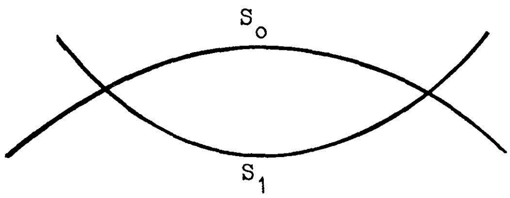


$r \geqq 3$,

$r = 5$ in this example

For $r = 2$ the configuration looks as follows:

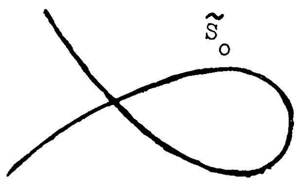
(17)



There are two transversal intersections of S_0 and S_1.

If $r = 1$, there is a special situation because the curves S_0 and S_1 of Y^+ intersect transversally in one point and S_0 and S_1 become identified under V. Thus under the map $Y^+ \to Y^+/V$ the string of rational curves S_k is mapped onto one rational curve $\tilde{S}_0$ in Y^+/V with one ordinary double point (which was previously also denoted by S_0, but must here be distinguished).

(18)



$r = 1$

Lemma. For $r = 1$ *we have in* Y^+/V

$$-\tilde{S}_0 \cdot \tilde{S}_0 = b_0 - 2$$

Proof. Let c_1 and $\tilde{c}_1$ denote the first Chern classes of Y^+ and Y^+/V respectively. Let π be the map $Y^+ \to Y^+/V$. Then $\pi^*\tilde{c}_1 = c_1$ and

$$\tilde{c}_1(\tilde{S}_0) = c_1(S_0)$$

where we evaluated the first Chern classes on the cycles $\tilde{S}_0$ and S_0. By the adjunction formula (0.6)

$$c_1(S_0) - S_0 \cdot S_0 = 2$$
$$c_1(\tilde{S}_0) - \tilde{S}_0 \cdot \tilde{S}_0 + 2 = 2.$$

The summand 2 on the left side of the second formula is the contribution of the double point of $\tilde{S}_0$ in the adjunction formula. We get

$$\tilde{S}_0 \cdot \tilde{S}_0 = S_0 \cdot S_0 + 2 = -b_0 + 2$$

which completes the proof.

By $((b_0, b_1, \ldots, b_{r-1}))$ we denote a *cycle* of numbers. (A cycle is given by an ordered set of r numbers. Two ordered sets are identified if they can be obtained from each other by a cyclic permutation.)

For any cycle $((b_0, b_1, \ldots, b_{r-1}))$ of natural numbers $\geqq 2$ (at least one $b_j \geqq 3$) we have constructed a complex manifold Y^+/V which we shall denote now by $Y((b_0, \ldots, b_{r-1}))$.

In this complex manifold of complex dimension 2 (we shall often say "complex surface") we have a configuration (16), (17) or (18) of rational curves. The corresponding matrices of intersection numbers are

$$\begin{bmatrix} -b_0 & 1 & 0 & \ldots & 0 & 1 \\ 1 & -b_1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & \ldots & \ldots & .. \\ .. & \ldots & \ldots & \ldots & \ldots & .. \\ 0 & \ldots & \ldots & 1 & -b_{r-2} & 1 \\ 1 & 0 & \ldots & 0 & 1 & -b_{r-1} \end{bmatrix} \qquad \text{for } r \geqq 3$$

and

$$\begin{pmatrix} -b_0 & 2 \\ 2 & -b_1 \end{pmatrix} \qquad \text{for } r = 2$$

By the lemma we have for $r = 1$ the 1×1-matrix $(-b_0+2)$. It is easy to show that these matrices are negative definite in all cases.

If all the b_i of a cycle equal 2, then the matrix is negative semi-definite with a null-space of dimension 1. Thus to get negative definiteness we do need the assumption $b_j \geqq 3$ for at least one j.

The negative definiteness implies, according to Grauert [17], that the configurations (16), (17) or (18) can be blown down to give an isolated normal point P in a complex space $\overline{Y}((b_0, \ldots, b_{r-1})$. We have a holomorphic map

$$\sigma : Y((b_0, \ldots, b_{r-1})) \to \overline{Y}((b_0, \ldots, b_{r-1}))$$

with

$$\sigma\Big(\bigcup_{k=0}^{r-1} S_k\Big) = P$$

The map

$$\sigma : Y((b_0, \ldots, b_{r-1})) - \bigcup_{k=0}^{r-1} S_k \to \overline{Y}((b_0, \ldots, b_{r-1})) - \{P\}$$

is biholomorphic. The configurations (16), (17), (18) represent the unique minimal resolution of the point P, because they do not contain exceptional curves of the first kind, i.e. non-singular rational curves of selfintersection number -1. *Thus the point P is singular.*

The first lemma of 2.3 shows that we have a natural map

$$Y((b_0, \ldots, b_{r-1})) \to \overline{\mathfrak{H}^2/G(M, V)}$$

and a commutative diagram

$$\begin{array}{ccc} Y((b_0, \ldots, b_{r-1})) & \to & \overline{\mathfrak{H}^2/G(M, V)} \\ \downarrow \sigma & \nearrow \tilde{\sigma} & \\ \overline{Y}((b_0, \ldots, b_{r-1})) & & \end{array}$$

where $\tilde{\sigma}$ is biholomorphic and $\tilde{\sigma}(P) = \infty$ (in the notation of 2.1). The map $\tilde{\sigma}$ is biholomorphic also in P because one can introduce at most one normal complex structure in $\overline{\mathfrak{H}^2/G(M, V)}$ extending the complex structure of $\mathfrak{H}^2/G(M, V)$.

We have established the existence of the normal complex space $\overline{\mathfrak{H}^2/G(M, V)}$ directly without using the Proposition given in 2.1. We need only *define* $\tilde{\sigma}$ to be biholomorphic. Also we have given the resolution of the singular point ∞ which was added to $\mathfrak{H}^2/G(M, V)$. We summarize our results:

THEOREM. *Let* $((b_0, b_1, \ldots, b_{r-1}))$ *be a cycle of natural numbers* $\geqq 2$ *(at least one* $b_j \geqq 3$*). Put*

$$w_0 = [[b_0, \ldots, b_{r-1}, b_0, \ldots, b_{r-1}, \ldots]] = [[\overline{b_0, \ldots, b_{r-1}}]]$$

(infinite periodic continued fraction). Then $K = \mathbf{Q}(w_0)$ *is a real quadratic field and* $M = \mathbf{Z}w_0 + \mathbf{Z}\,.\,1$ *a complete* $\mathbf{Z}$*-module of* K*. The cycle* $((b_0, \ldots, b_{r-1}))$ *determines a totally positive unit* A_r *of* K *with* $A_r M = M$*. The unit* A_r *generates an infinite cyclic subgroup* V *of* U_M^+*, the group of all totally positive units* ε *of* K *with* $\varepsilon M = M$*. The unique singular point* ∞ *of* $\overline{\mathfrak{H}^2/G(M, V)}$*, where* $G(M, V)$ *is the natural semi-direct product of* M *and* V*, admits a cyclic resolution by rational curves* S_k *(configuration* (16), (17) *or* (18)*) with selfintersection numbers* $S_k \cdot S_k = -b_k$ *(for* $r = 1$ *we have* $\tilde{S}_0 \cdot \tilde{S}_0 = -b_0 + 2$*). This resolution is given by the complex surface* $Y((b_0, \ldots, b_{r-1}))$ *which we canonically associated to a cycle.*

Remark 1. Laufer [50] has shown that two normal singular points (in complex dimension 2) which admit a resolution with a given cyclic configuration of rational curves of type (16), (17) or (18) and given self-intersection numbers are isomorphic. Hence the singularity P of $Y((b_0, \ldots, b_{r-1}))$ which we have constructed is up to isomorphism the unique singularity with the given cyclic configuration of rational curves and the given selfintersection numbers. (These singularities are called cyclic singularities.) Reversal of the cycle gives an isomorphic singularity.

Remark 2. The construction of Y in 2.2 applies also to the case where all b_k equal 2. Then we have $u_j \cdot v_j = u_k \cdot v_k$ (compare (5)) and hence obtain a holomorphic function $f : Y \to \mathbf{C}$. As in 2.3, we have a properly discontinuous action of an infinite cyclic group V on $Y^\varepsilon = \{p \mid p \in Y, |f(p)| < \varepsilon\}$, for ε positive and sufficiently small, whose generator maps the curve S_k to S_{k+r}. The period $r \geqq 1$ can be choosen arbitrarily.

The function f is invariant under V; thus we get a holomorphic map

$$f : Y^\varepsilon / V \to \{ z \mid | z | < \varepsilon \}$$

All fibres of f are non-singular elliptic curves except $f^{-1}(0)$ which is a configuration of rational curves of type (16), (17), (18) where now all b_k equal 2. The fibring we have constructed is of type ${}_1I_r$ in the sense of Kodaira [45], Part II. We have seen:

Cycles $((2, ..., 2))$ *give an infinite continued fraction of value* 1 *and correspond to an elliptic fibring. Cycles* $((b_0, ..., b_{r-1}))$, $(b_k \geqq 2$, *at least one* $b_j \geqq 3)$, *give an infinite continued fraction whose value is a quadratic irrationality. These cycles determine singular points.*

2.5. The theorem in 2.4 actually provides a *resolution* of the singular point of $\overline{\mathfrak{H}^2/G(M, V)}$ (see 2.1 with $n = 2$) for any complete **Z**-module M of a real quadratic field K and an infinite cyclic subgroup V of U_M^+ of any given index $a = [U_M^+ : V]$. We need a lemma.

Lemma. Consider the **Z**-*module* M *defined by a periodic function* $k \mapsto b_k \geqq 2$ *(with* $b_j \geqq 3$ *for at least one* j*). Let* $r \geqq 1$ *be the smallest period. Then* A_r *(see* 2.3*) is a generator of* U_M^+.

Proof. We shall denote ordinary continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

by $[a_0, a_1, a_2, ...]$. The relation between the two types of continued fractions is as follows:

$$[a_0, a_1, z] = [[a_0 + 1, \underbrace{2, \ldots, 2}_{a_1 - 1}, z + 1]] \tag{19}$$

where z is an indeterminante and a_1 a natural number $\geqq 1$. Using (19) the lemma can be derived from similar results for ordinary continued fractions (compare [6], Kap. II, § 7). A proof is also given in [12]. Another proof was communicated to the author by J. Rohlfs.

Two complete **Z**-modules M_1, M_2 of the same real quadratic field K are *strictly equivalent* (2.1) if there exists a totally positive number $\alpha \in K$ with $\alpha M_1 = M_2$. We have $U_{M_1}^+ = U_{M_2}^+$.

The actions of $G(M_1, V)$ and $G(M_2, V)$ on $\mathfrak{H}^2$ are equivalent under the automorphism $(z_1, z_2) \mapsto (\alpha z_1, \alpha z_2)$ of $\mathfrak{H}^2$. Any module M_1 is strictly equivalent to a module of the form $M_2 = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$ where $w_0 \in K$ and $0 < w_0' < 1 < w_0$. (This is easy to prove, as was shown to me by H. Cohn.) Then the continued fraction $w_0 = [[b_0, b_1, ...]]$ is purely periodic, i.e. periodicity starts with b_0. This can be proved in the same way as an analogous result for ordinary continued fractions ([60], § 22). Let r be the smallest period. We can resolve the singularity of $\overline{\mathfrak{H}^2/G(M_2, U_{M_2}^+)}$ by the method of 2.3 and 2.4, since by the preceding lemma $U_{M_2}^+ = \{(A_r)^n \mid n \in \mathbf{Z}\}$. The resolution is described by the primitive cycle $((b_0, ..., b_{r-1}))$ where primitive means that the cycle cannot be written as an "unramified covering" of degree > 1. The cycle $((2, 3, 5, 2, 3, 5)) = ((2, 3, 5))^2$ is not primitive, for example.

For any primitive cycle $((b_0, ..., b_{r-1}))$ we obtain a module $\mathbf{Z}w_0 + \mathbf{Z} \cdot 1$ with $w_0 = [[b_0, b_1, ...]]$. In the cycle we must allow cyclic permutations. This changes the module to a module $\mathbf{Z}w_k + \mathbf{Z} \cdot 1$ (see 2.3). But $\mathbf{Z}w_0 + \mathbf{Z} \cdot 1 = \mathbf{Z}A_{k-1} + \mathbf{Z}A_k$ and $A_{k-1}/A_k = w_k$, where A_k is totally positive (see 2.3). Therefore, the strict equivalence class of the module only depends on the cycle. *If one reverses the order (orientation) of the cycle, the associated equivalence class of modules is replaced by the conjugate one* (see (13)).

If we start from a strict equivalence class of modules, it determines, as explained above, an isomorphism class of singularities (represented by the singularity of $\mathfrak{H}^2/G(M_2, U_{M_2}^+)$).

But isomorphic singularities must give the same unoriented cycle in their canonical minimal resolutions. "Unoriented" means that we cannot distinguish between $((b_0, ..., b_{r-1}))$ and $((b_{r-1}, ..., b_0))$. But, in fact, if we represent the class of modules as above by $M_2 = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$, then the cycle of w_0 is uniquely determined including the orientation. If this were not the case, it would follow that M_2 and M_2' are strictly equivalent. Then the singularity and its resolution admit an involution showing that the cycles $((b_0, ..., b_{r-1}))$ and $((b_{r-1}, ..., b_0))$ are equal. (Details are left to the reader. The relation between strict equivalence classes of modules and primitive cycles can be derived, of course, also without using the resolution, compare 2.6.)

We have established a bijective map between primitive admissible cycles (all $b_k \geqq 2$ and at least one $b_j \geqq 3$) and the strict equivalence classes of complete $\mathbf{Z}$-modules (where the real quadratic field K varies).

The preceding discussion yields the following theorem.

THEOREM. *Let K be a real quadratic field and M a complete $\mathbf{Z}$-module in K. Let $((b_0, b_1, ..., b_{r-1}))$ be the primitive cycle belonging to M. Let V be the subgroup of U_M^+ of index a. Then the resolution of the singular point of $\overline{\mathfrak{H}^2/G(M, V)}$ is given by the cycle $((b_0, b_1, ..., b_{r-1}))^a$.*

Remark. The structure of the local ring $\mathfrak{O}(M, V)$ at the point ∞ of $\mathfrak{H}^2/G(M, V)$ was described in 2.1. For any admissible cycle $((b_0, \ldots, b_{r-1}))$, not necessarily primitive, the functions $f \in \mathfrak{O}(M, V)$ can be written as power series' in u_0, v_0 where u_0, v_0 is the coordinate system of 2.3 (11) with $A_0 = 1$ and $A_{-1} = w_0 = [[b_0, \ldots, b_{r-1}]]$. We could use also any other coordinate system u_k, v_k.

Let $(u_0, v_0)^n = u_0^{n1} \cdot v_0^{n2}$ for $n = (n_1, n_2)$ and

$$Tn = \begin{pmatrix} -q_{r-1} & p_{r-1} \\ -q_r & p_r \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \text{ see 2.2 (7),}$$

then $\mathfrak{O}(M, V)$ is the ring of all power series'

$$f = a_0 + \sum_n a_n (u_0, v_0)^n$$

where the summation extends over all pairs $n = (n_1, n_2)$ of positive integers with $w_0' \leqq n_1/n_2 \leqq w_0$, the coefficients satisfy $a_{Tn} = a_n$, and the power series converges for

$$w_0 \log |u_0| + \log |v_0| < 0, \; w_0' \log |u_0| + \log |v_0| < 0$$

$$(w_0 \log |u_0| + \log |v_0|) \cdot (w_0' \log |u_0| + \log |v_0|) > \frac{1}{c},$$

(the positive constant c depending on f).

Observe that T (as fractional linear transformation) maps the intervall $[w_0', w_0]$ bijectively onto itself ($Tw_0' = w_0'$, $Tw_0 = w_0$). We have $Tx < x$ for $w_0' < x < w_0$ and therefore

$$\lim_{k \to \infty} T^k x = w_0' \text{ (for } w_0' \leqq x < w_0\text{) and } \lim_{k \to -\infty} T^k x = w_0 \text{ (for } w_0' < x \leqq w_0\text{)}$$

Example. Consider the Fibonacci numbers

$$\ldots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13$$

where $F_0 = 0$, $F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ ($k \in \mathbf{Z}$). The numbers $G_k = F_{2k+1}$ ($k \in \mathbf{Z}$) are all positive and satisfy $G_{k+1} = 3G_k - G_{k-1}$. The function

$$f(u_0, v_0) = \sum_{k=-\infty}^{\infty} u_0^{G_{k-1}} \cdot v_0^{G_k}$$

represents an element of $\mathfrak{D}(M, U_M^+)$ where

$$M = \mathbf{Z}w_0 + \mathbf{Z} \quad \text{and} \quad w_0 = [[\overline{3}]] = \tfrac{1}{2}(3+\sqrt{5}).$$

2.6. The primitive cycle associated to a module M can be found also without using a base w_0, 1 of M with $0 < w_0' < 1 < w_0$: Real numbers x, y are called strictly equivalent if there exists an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbf{Z})$ such that

$$y = \frac{ax + b}{cx + d}.$$

Any irrational number x has a unique infinite continued fraction development

$$x = [[a_0, a_1, a_2, \ldots]]$$

where $a_i \in \mathbf{Z}$ and $a_i \geqq 2$ for $i \geqq 1$ and where $a_i \geqq 3$ for infinitely many indices i. Two irrational numbers are strictly equivalent if and only if their continued fractions $[[a_0, a_1, \ldots]]$ and $[[a_0', a_1', \ldots]]$ coincide from certain points on, i.e. $a_{j+i} = a_{k+i}'$ for some j and k and for all $i \geqq 0$. This is analogous to a classical result on ordinary continued fractions ([60], Satz 2.24).

A quadratic irrationality w admits a continued fraction which is periodic from a certain point on. It is purely periodic if and only if $0 < w' < 1 < w$ as mentioned before. The periodicity of the continued fraction of w determines a primitive cycle $((b_0, \ldots, b_{r-1}))$ which is admissible (all $b_i \geqq 2$, at least one $b_j \geqq 3$). Thus two quadratic irrationalities are strictly equivalent if and only if their cycles agree, and *we have a bijection between strict equivalence classes of quadratic irrationalities and admissible primitive cycles. The admissible primitive cycles are in one-to-one correspondence with the strict equivalence classes of complete* **Z***-modules in real quadratic fields K where K varies* (see 2.5).

A complete **Z**-module M of a real quadratic field K will be oriented by using the admissible bases (β_1, β_2) of M with $\beta_1\beta_2' - \beta_2\beta_1' > 0$. By restricting the norm function $(N(x) = xx'$ for $x \in K)$ to M we obtain an indefinite quadratic form f on M with rational values. The exists a unique positive rational number m such that $m \cdot f$ is integral and with respect to an admissible base of M can be written as

$$au^2 + buv + cv^2$$

where $a, b, c \in \mathbf{Z}$ and $(a, b, c) = 1$. The pairs (u, v) are in $\mathbf{Z} \oplus \mathbf{Z} \cong M$. The discriminant $D_M = b^2 - 4ac$ is positive and not a square number.

In this way, we get a *bijection between strict equivalence classes of complete* **Z***-modules of real quadratic fields and the isomorphism classes under* $\mathbf{SL}_2(\mathbf{Z})$ *of integral indefinite primitive binary quadratic forms of non-square discriminant.*

Remark. The discriminant D of such a quadratic form can be written uniquely as

$$D = D_K \cdot f^2, \quad f \geqq 1,$$

where D_K is the discriminant of the real quadratic field $K = \mathbf{Q}(\sqrt{D})$. Then the corresponding strict equivalence class of modules can be represented by an ideal in the order (subring of $\mathfrak{o}_K$) which as an additive group has index f in $\mathfrak{o}_K$, and this is the smallest f such that the equivalence class of M can be represented in this way.

The strict equivalence class of the "first root"

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{where} \quad \sqrt{b^2 - 4ac} > 0$$

depends only on the equivalence class of the quadratic form.

We obtain a bijection between $\mathbf{SL}_2(\mathbf{Z})$*-equivalence classes of integral indefinite primitive binary quadratic forms of non-square discriminant and strict equivalence classes of quadratic irrationalities.*

All the bijections are compatible with each other as can be checked easily. Let us collect the bijections:

strict equivalence classes of complete **Z**-modules in real quadratic fields $\leftrightarrow$

admissible primitive cycles of natural numbers $\leftrightarrow$

strict equivalence classes of quadratic irrationalities $\leftrightarrow$

$\mathbf{SL}_2(\mathbf{Z})$-equivalence classes of integral indefinite primitive binary quadratic forms of non-square discriminant $\leftrightarrow$

isomorphism classes of cyclic singularities with a primitive cycle and as additional structure a prefered orientation of the cycle (compare 2.4, Remark 1).

Example. Let d be a square-free number > 1 and suppose $d \equiv 2 \bmod 4$ or $d \equiv 3 \bmod 4$. The $(\sqrt{d}, 1)$ is an admissible $\mathbf{Z}$-base of the ideal (1) in $\mathfrak{o}_K$ for $K = \mathbf{Q}(\sqrt{d})$. The quadratic form is given by

$$- u^2 d + v^2$$

and has discriminant $4d$. The first root equals $\dfrac{-\sqrt{d}}{d} = -\dfrac{1}{\sqrt{d}}$ which is equivalent to $\sqrt{d}$. (Take always the positive square root). The admissible cycle of natural numbers is obtained by developing $\sqrt{d}$ in a continued fraction.

§ 3. Numerical invariants of singularities and of Hilbert modular surfaces

3.1. Let X be a compact oriented manifold of dimension $4k$ with or without boundary. Then $H^{2k}(X, \partial X; \mathbf{R})$ is a finite dimensional real vector space over which we have a bilinear symmetric form B with

$$B(x, y) = (x \cup y)[X, \partial X], \quad \text{for } x, y \in H^{2k}(X, \partial X; \mathbf{R}),$$

where $[X, \partial X]$ denotes the generator of $H_{4k}(X, \partial X; \mathbf{Z})$ defined by the orientation. The signature of B, i.e., the number of positive entries minus the number of negative entries in a diagonalized version, is called sign (X). If X has no boundary and is differentiable, then according to the signature theorem ([36], p. 86)

$$\text{sign}(X) = L_k(p_1, \ldots, p_k)[X], \tag{1}$$

where L_k is a certain polynomial of weight k in the Pontrjagin classes of X with rational coefficients $\left(p_j \in H^{4j}(X, \mathbf{Z})\right)$.

Let N be a compact oriented differentiable manifold without boundary of dimension $4k - 1$ together with a given trivialization α of its stable tangent bundle. (Such a trivialization need not exist). We shall associate to the pair (N, α) a rational number $\delta(N, \alpha)$. Since N has a trivial stable tangent bundle, all its Pontrjagin and Stiefel-Whitney numbers vanish. Therefore N bounds a $4k$-dimensional compact oriented differentiable manifold X. By the parallelization α we get from the stable tangent bundle of X an $\mathbf{SO}$-bundle over X/N. We denote its Pontrjagin classes by