

# 5. Conclusion

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The discriminant is  $4p^2(p - \alpha^2 r^2)$  which cannot vanish, so that, as before, the first factor in (10) must be zero, and we have

$$(11) \quad (\alpha^2 + \beta^2)r^2 - p = 0$$

which is a contradiction since  $\alpha^2 + \beta^2 > 1$  and we are supposing that  $|r| > 1$ .

Therefore we cannot have  $|r| > 1$ ,  $K \neq 0$ , and  $L \neq 0$ . If  $|r| = 1$  we see that  $K = L = 0$  since  $|x - \alpha r| < p$  and  $|y - \beta r| < p$  in this case. If  $|r| > 1$  with  $K = L = 0$  we would have  $x = \alpha r$ ,  $y = \beta r$  and hence  $(x, y) > 1$ , whereas  $x$  and  $y$  are relatively prime. Finally it remains to consider the possibility of having  $|r| > 1$  with one of  $K$  and  $L$  zero, the other nonzero. This if we suppose that  $|r| > 1$ ,  $K = 0$ ,  $L \neq 0$ , we obtain (9) which, as we have seen, leads to a contradiction. On the other hand the supposition that  $|r| > 1$  with  $K \neq 0$ ,  $L = 0$  implies that (11) would hold with  $r^2 > 1$ .

We conclude that  $|r| = 1$ ,  $K = 0$  and  $L = 0$ . Hence  $x = \pm \alpha$ ,  $y = \pm \beta$  and  $\alpha^2 + \beta^2 = p$ .

In [1], Corollary 2, we observed that if  $p = x^2 + y^2$  then, in our notation,  $y$  is a quadratic residue of  $p$ . Collecting our results we have the

COROLLARY. Let  $p = x^2 + y^2$  where  $p$  is a prime of the form  $4n + 1$  with  $x$  and  $y$  given by (3) and (4). Then  $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$  and  $\left(\frac{y}{p}\right) = 1$ .

## 5. CONCLUSION

We saw that  $x = \pm \alpha$ ,  $y = \pm \beta$ . When  $p = 13$  we have  $y = -3$ ,  $\beta = -3$ ; when  $p = 29$ ,  $y = -5$ ,  $\beta = 5$ , and when  $p = 41$ ,  $y = 5$ ,  $\beta = 5$ . Hence the sign of  $y$ , determined by the approximants to a continued fraction depends on the integer  $m$ , the number of terms in the finite segment of (2) which is used, can agree with that of  $\beta$  or be opposite that of  $\beta$ . The same applies to  $x$  and  $\alpha$ . In [1], Theorem 1, we gave a construction which always gives positive values for  $x$  and  $y$ . Other various constructions, as we have seen, do not have this property.

Finally we comment on the numbers  $\frac{(2n)!}{2(n!)^2}$  which we denote by  $a_n$  for  $n = 1, 2, 3, \dots$

The members of the sequence  $\{a_n\}$  are related to the numbers  $b_{n+1} = \frac{(2n)!}{(n+1)!n!}$ ,  $n = 0, 1, 2, \dots$ , which, as mentioned by Becker [2], have a variety of applications. Birkhoff [3] pointed out that  $b_n$  is an integer for every positive integer  $n$ , and noted the recurrence relation  $b_n = \sum_{i=1}^{n-1} b_i b_{n-i}$ ; a relation which was also obtained by Wedderburn [10].

The results of this note depend on the fact that  $a_n$  is an integer, at least when  $p = 4n + 1$  is a prime. Although it is known that  $a_n$  is an integer for every positive integer  $n$ , we can see that this also follows readily from [3]. For we have  $2a_n = (n+1)b_{n+1}$ . If  $n$  is even, it follows that  $b_{n+1}$  is even since  $(2, n+1) = 1$ . Therefore  $a_n = (n+1) \frac{b_{n+1}}{2}$  is an integer. If  $n$  is odd then  $2 \mid (n+1)$  and in this case also  $a_n = \frac{n+1}{2} b_{n+1}$  is an integer. A list of values for  $a_n$  can be obtained from the second column of a table in [2], page 699, headed  $N_n$ , by multiplying the  $(n+1)$ st member by  $\frac{n+1}{2}$ .

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