2. Ordered Fields Which Properly Contain the Reals

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$$k > \frac{1}{|s - |\alpha||}$$

and from this we get

$$s - |\alpha| > \frac{1}{k} \quad or \quad |\alpha| - s > \frac{1}{k}.$$

Case 1. $s - |\alpha| > \frac{1}{k}$. In this case $s - \frac{1}{k} > |\alpha|$, but then by definition of A we see that $s - \frac{1}{k}$ is a real upper bound of A. Moreover, $s - \frac{1}{k}$ is smaller than the *least* upper bound s, which is absurd.

Case 2. $|\alpha| - s > \frac{1}{k}$. Then $|\alpha| > s + \frac{1}{k}$, so $s + \frac{1}{k} \in A$ by definition of A. But s is an upper bound for A so $s + \frac{1}{k} \leqq s$ from which follows $k \leqq 0$; but this contradicts the fact that k is a natural number.

(Q.E.D.)

2. Ordered Fields Which Properly Contain the Reals

In this section we shall assume that F is an ordered field which has the real numbers R as a proper subordered field. We have already seen that F must be non-Archimedean. N will be used to denote the set of natural numbers.

An element $a \in F$ is said to be

infinitesimal if $|a| < r$ for each positive real r.

finite if $|a| \leqq r$ for some real r.

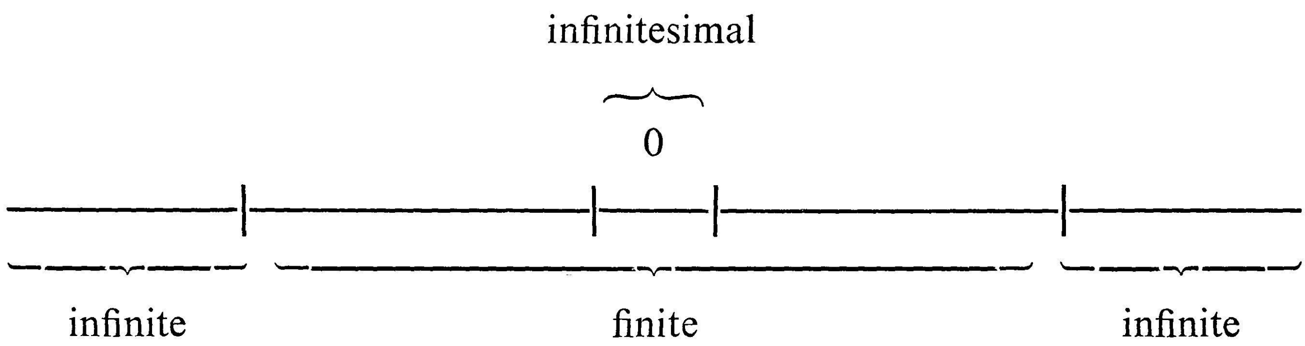
infinite if $|a| > r$ for every real r.

The number 0 is certainly infinitesimal, but it is easy to see that there are also non-zero infinitesimals and infinites as follows:

F being non-Archimedean must contain an element b such that $n \leqq b$ for all $n \in N$. This implies that $n < b$ all $n \in N$ and, in fact, $r < b$ all $r \in R$. Thus b is infinite and $\frac{1}{b}$ is infinitesimal.

Whenever α and β differ by an infinitesimal we say that α is *near* β or α is *infinitely close* to β. We symbolize this $\alpha \approx \beta$. The relation $\approx$ is easily seen to be an equivalence relation. The infinitesimals are precisely those numbers which are infinitely close to 0.

A crude picture of F appears below.



Note that infinitesimals are finite, and reals are finite; in fact, all numbers of the form real + infinitesimal are finite. It turns out that the finite numbers can always be written this way and we formulate this as a theorem below.

THEOREM 2.1. Every finite number can be written in the form

real + infinitesimal.

Moreover, this representation is unique. Put differently, every finite number is infinitely close to a uniquely determined real number.

PROOF:

Uniqueness of representation—suppose that

$$r_1 + \varepsilon_1 = r_2 + \varepsilon_2$$

where $r_1, r_2 \in R$ and $\varepsilon_1, \varepsilon_2$ are infinitesimal. Then

$$r_1 - r_2 = \varepsilon_2 - \varepsilon_1.$$

Now the left side is real and it is easy to see that the right side is infinitesimal. But the only real which is infinitesimal is zero, so both sides are zero; thus

$$r_1 = r_2 \text{ and } \varepsilon_1 = \varepsilon_2.$$

Existence of representation—let α be finite, to show that $\alpha = a + \varepsilon$ where a is real and ε is infinitesimal. Let $A = \{ x \in R \mid x < \alpha \}$. Since α is finite there exists $r \in R$ such that $\alpha < r$. Now A is bounded by r, so it has a real least upper bound a.

Case 1. $|\alpha - a| = s$ for some $s \in R$. Then α has the desired representation $\alpha = (a \pm s) + 0$.

Case 2. $|\alpha - a|$ is not real. Now assert (and will show) that $\alpha - a$ is infinitesimal; then α would have the desired representation $\alpha = a + (\alpha - a)$. We show that $\alpha - a$ is infinitesimal by contradiction. Suppose not, then $|\alpha - a| \geqq s > 0$ for some $s \in R$.

Case 2.1. $a - \alpha > s$. Then $a - s > \alpha$ so $a - s$ is a real upper bound for A. But a is the least upper bound, so $a \leqq a - s$. From this follows $0 \geqq s$ which contradicts s being positive.

Case 2.2. $\alpha - a > s$. Then $\alpha > a + s$, so $a + s \in A$. But a is an upper bound for A, so $a \geqq a + s$. Thus $0 \geqq s$ which contradicts s being positive.

(Q.E.D.)

The following rules are all easy to verify:

$$\text{finite} \pm \text{finite} = \text{finite}$$

$$\text{infinitesimal} \pm \text{infinitesimal} = \text{infinitesimal}$$

$$\text{finite} \pm \text{infinite} = \text{infinite}$$

$$\text{finite} \cdot \text{finite} = \text{finite}$$

$$\text{infinitesimal} \cdot \text{finite} = \text{infinitesimal}$$

$$\text{infinite} \cdot \text{infinite} = \text{infinite}$$

$$\frac{1}{\text{infinite}} = \text{infinitesimal}$$

$$\frac{1}{\text{non-zero infinitesimal}} = \text{infinite}$$

These rules tell us, among other things, that the finite numbers constitute a ring and the infinitesimals constitute an ideal in this ring.

The situations below are indeterminate:

$$\text{infinite} \pm \text{infinite} = ?$$

$$\text{finite} \cdot \text{infinite} = ?$$

$$\frac{\text{infinitesimal}}{\text{infinitesimal}} = ?$$

$$\frac{\text{infinite}}{\text{infinite}} = ?$$

$$\frac{\text{finite}}{\text{finite}} = ?$$

Note that since we are working in a field, $0 \cdot$ infinite is actually defined and has the value 0.

The infinitely close symbol $\approx$ can be manipulated as follows:

If $\alpha \approx \beta$ and $u \approx v$, then

$$\alpha + u \approx \beta + v$$

$$\alpha - u \approx \beta - v$$

$$\alpha \cdot u \approx \beta \cdot v \quad \text{(provided } \alpha, u \text{ are finite)}$$

$$\frac{\alpha}{u} \approx \frac{\beta}{v} \quad \text{(provided } \alpha \text{ is finite and } u \text{ ins't infinitesimal)}$$

The hint to proving the last two is to write $\alpha = \beta + \varepsilon$, and $u = v + \delta$ where ε, δ are infinitesimals.

We have already seen that every finite number α is infinitely close to a uniquely determined real number; we call this real number the *standard part* of α and denote it by $^\circ\alpha$.

The following are obtained from the rules just given (α, β are assumed to be finite):

$$^\circ(\alpha+\beta) = {}^\circ\alpha + {}^\circ\beta$$

$$^\circ(\alpha-\beta) = {}^\circ\alpha - {}^\circ\beta$$

$$^\circ(\alpha \cdot \beta) = {}^\circ\alpha \cdot {}^\circ\beta$$

$$^\circ\!\left(\frac{\alpha}{\beta}\right) = \frac{^\circ\alpha}{^\circ\beta} \quad \text{(provided } \beta \text{ isn't infinitesimal).}$$

Remember now! The symbol $^\circ\alpha$ only makes sense when α is finite.