

# CONSTRUCTION OF GAUSS

Autor(en): **Barnes, C. W.**

Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **20 (1974)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

Persistenter Link: <https://doi.org/10.5169/seals-46891>

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

# A CONSTRUCTION OF GAUSS

by C. W. BARNES

## 1. INTRODUCTION

Every prime of the form  $4n + 1$  can be expressed uniquely as the sum of two squares. Suppose  $p = x^2 + y^2$  where  $p$  is a prime of the form  $4n + 1$ . A construction for  $x$  and  $y$  was given by Legendre [8] in terms of the continued fraction for  $\sqrt{p}$ . In [1] we gave a new construction for  $x$  and  $y$ , again using the continued fraction for  $\sqrt{p}$ . A summary of the various constructions is given in Davenport [5], pages 120-123.

Gauss [6] remarked that if  $p = 4n + 1$ , and if  $\alpha$  and  $\beta$  are defined by  $\beta \equiv \frac{(2n)!}{2(n!)^2} \pmod{p}$ ,  $\alpha \equiv (2n)! \beta \pmod{p}$ , where  $|\alpha| < \frac{p}{2}$ ,  $|\beta| < \frac{p}{2}$  then  $p = \alpha^2 + \beta^2$ ; a particularly simple construction to state. Proofs of the construction of Gauss were given by Cauchy [4], page 414, and Jacobsthal [7]; however, neither of them is simple.

In the present note we give a simple proof of the construction of Gauss based on the method in [1].

## 2. CONTINUED FRACTIONS

We continue with the notation in [1]. The results we need can be found in Perron [9]. We denote the simple continued fraction

$$(1) \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}}$$

by  $[a_0, a_1, \dots, a_n]$ . For  $0 \leq m \leq n$  we denote the numerator and denominator of the  $m^{\text{th}}$  approximant to  $[a_0, a_1, \dots, a_n]$  by  $A_m$  and  $B_m$  respectively.

If  $p$  is a prime of the form  $4n + 1$ , then

$$(2) \quad \sqrt{p} = [a_0, \overline{a_1, \dots, a_m, a_m, \dots, a_1}, 2a_0]$$

in the usual notation for a periodic continued fraction. The symmetric part of the period does not have a central term. In [1] we proved that  $p = x^2 + y^2$  where

$$(3) \quad x = pB_m B_{m-1} - A_m A_{m-1}$$

$$(4) \quad y = A_m^2 - pB_m^2$$

and where  $\frac{A_m}{B_m}$  is the  $m^{\text{th}}$  approximant to (2). We also showed that

$$(5) \quad p = \frac{A_m^2 + A_{m-1}^2}{B_m^2 + B_{m-1}^2}.$$

### 3. THE QUADRATIC CHARACTER OF

$$\frac{(2n)!}{2(n!)^2}.$$

It is well known that if  $p$  is a prime of the form  $4n + 1$  then  $\left\{ \left( \frac{p-1}{2} \right)! \right\}^2 \equiv -1 \pmod{p}$ ; that is,  $(2n)!^2 \equiv -1 \pmod{p}$ . We make use of this in the

LEMMA. If  $p = 4n + 1$  is a prime then  $\frac{(2n)!}{2(n!)^2}$  is a quadratic residue of  $p$ .

Proof. We use Euler's criterion. Thus if we suppose that  $\frac{(2n)!}{2(n!)^2}$  is a quadratic nonresidue of  $p$  we have  $\left\{ \frac{(2n)!}{2(n!)^2} \right\}^{\frac{p-1}{2}} \equiv -1 \pmod{p}$  and thus  $\left\{ (2n)!^2 \right\}^{\frac{p-1}{4}} \equiv - \left\{ 2(n!)^2 \right\}^{\frac{p-1}{2}} \pmod{p}$ . Since  $(2n)!^2 \equiv -1 \pmod{p}$  and  $n!^{p-1} \equiv 1 \pmod{p}$  we have  $(-1)^n \equiv -2 \frac{p-1}{2} \pmod{p}$ , or  $(-1)^{n+1} \equiv (-1)^{\frac{p^2+1}{8}}$ , using the standard result for the quadratic character of 2 with res-

pect to an odd prime. We finally get  $(-1)^{n+1} \equiv (-1)^{2n^2+n}$  or  $(-1)^{n+1} \equiv (-1)^n \pmod{p}$  which is a contradiction since  $p$  is an odd prime. Thus  $\frac{(2n)!}{2(n!)^2}$  is a quadratic residue of  $p$ .

#### 4. THE CONSTRUCTION OF GAUSS

**THEOREM.** Suppose  $p = 4n + 1$  is a prime and  $p = x^2 + y^2$  where  $x$  and  $y$  are given by (3) and (4). Let  $\beta$  and  $\alpha$  denote respectively the numerically smallest residues of  $\frac{(2n)!}{2(n!)^2}$  and  $(2n)! \beta$  modulo  $p$ , so that  $|\alpha| < \frac{p}{2}$ ,  $|\beta| < \frac{p}{2}$ . Then  $p = \alpha^2 + \beta^2$ .

*Proof.* By (5) we have, using the remark at the beginning of section 3,  $A_m^2 + A_{m-1}^2 \equiv 0 \pmod{p}$  and hence  $-A_m^2 \equiv A_{m-1}^2 \pmod{p}$ , so that  $\{(2n)!\}^2 A_m^2 \equiv A_{m-1}^2 \pmod{p}$ , and since  $p$  is a prime  $(2n)! A_m \equiv \pm A_{m-1} \pmod{p}$ . Supposing the negative sign holds we have  $(2n)! A_m^2 \equiv -A_m A_{m-1} \pmod{p}$ . Therefore we obtain  $(2n)! A_m^2 - (2n)! p B_m^2 \equiv (p B_m B_{m-1} - A_m A_{m-1}) \pmod{p}$ , so that by (3) and (4) we get

$$(6) \quad x \equiv (2n)! y \pmod{p}.$$

If the positive sign holds above it follows that  $x \equiv -(2n)! y \pmod{p}$  which is just as good for our present purposes since we are not concerned with the signs of  $x$  and  $y$ . We will comment on the signs in section 5.

By the lemma we have  $\left\{ \frac{(2n)!}{2(n!)^2} \right\}^{\frac{p-1}{2}} \equiv 1 \pmod{p}$  so  $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} (n!)^{p-1} \pmod{p}$ , and therefore  $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} \pmod{p}$  since  $(n!, p) = 1$ . We have  $x \equiv \pm (2n)! y \pmod{p}$ , and since each of  $y$  and  $-1$  is a quadratic residue of  $p$ ,  $x^{\frac{p-1}{2}} \equiv (2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}} \pmod{p}$ , and in terms of the Legendre symbol it follows that  $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$ ; that is, the quadratic character of  $x$  with respect to  $p$  is the same as the quadratic character of 2 with respect to  $p$ .

Suppose 2 is a quadratic residue of  $p$ . Then

$$2^{\frac{p-1}{2}} (n!)^{p-1} (A_m A_{m-1})^{\frac{p-1}{2}} \equiv (A_m A_{m-1})^{\frac{p-1}{2}} \equiv (-x)^{\frac{p-1}{2}} \equiv x^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$$

Next, if 2 is a quadratic nonresidue of  $p$  we have

$$2^{\frac{p-1}{2}} (n!)^{p-1} (A_m A_{m-1})^{\frac{p-1}{2}} \equiv -(-x)^{\frac{p-1}{2}} \equiv -(x)^{\frac{p-1}{2}} \equiv -(-1) \equiv 1 \pmod{p},$$

and we conclude that  $2(n!)^2 A_m A_{m-1}$  is a quadratic residue of  $p$ . By (3), (4), and (6) we have

$$(2n)! y \equiv -A_m A_{m-1} \pmod{p},$$

$$2(n!)^2 (2n)! y \equiv -2(n!)^2 A_m A_{m-1} \pmod{p}$$

and

$$-2(n!)^2 (2n)! y \equiv b^2 \pmod{p}$$

for some quadratic residue  $b^2$ . Therefore

$$-2(n!)^2 (2n)! y \equiv -(2n)!^2 b^2 \pmod{p},$$

$$-2(n!)^2 y \equiv -(2n)! b^2 \pmod{p},$$

and finally

$$y \equiv \frac{(2n)!}{2(n!)^2} b^2 \pmod{p}.$$

Hence by (6)

$$x \equiv \frac{(2n)!^2}{2(n!)^2} b^2 \pmod{p}.$$

Let  $b^2 \equiv r \pmod{p}$ ,  $|r| < \frac{p}{2}$ , so that  $(r, p) = 1$ . Then in terms of  $\alpha$ ,

$\beta$ , and  $r$ ,  $x \equiv \alpha r \pmod{p}$  and  $y \equiv \beta r \pmod{p}$ . There are unique integers  $K$  and  $L$  such that  $x = \alpha r + Kp$ ,  $y = \beta r + Lp$ . Then

$$x^2 + y^2 = (\alpha^2 + \beta^2) r^2 + (K^2 + L^2) p^2 + 2rp(\alpha K + \beta L),$$

or

$$p = (\alpha^2 + \beta^2) r^2 + (K^2 + L^2) p^2 + 2rp(\alpha K + \beta L).$$

Suppose that  $|r| > 1$ ,  $K \neq 0$ , and  $L \neq 0$ . The last equation can be written

$$(7) \quad pK^2 + (2r\alpha p)K + \{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\} = 0.$$

Since (7) is a quadratic in  $K$  and we are supposing that the integral root is not zero we have

$$K \mid \{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\}.$$

There is an integer  $t$  such that

$$L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p = Kt$$

and therefore (7) vanishes when

$$K = \frac{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p}{t}.$$

That is

$$(8) \quad \{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\} \{t^2 + 2r\alpha pt + p\{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\}\} = 0$$

The discriminant of the quadratic function

$$t^2 + 2r\alpha pt + p\{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\}$$

is  $4p^2\{p - (pL + \beta r)^2\}$  which is not zero. It follows that the second factor in (8) cannot be zero; otherwise we would have two distinct integral values for  $t$  giving rise to two distinct integers  $K$ , whereas  $K$  is unique. Hence we have

$$(9) \quad p^2L^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p = 0$$

and since we are supposing that  $L \neq 0$ , we see that

$L \mid \{(\alpha^2 + \beta^2)r^2 - p\}$  so that for an integer  $u$  we have  $(\alpha^2 + \beta^2)r^2 - p = Lu$  and (9) vanishes when

$$L = \frac{(\alpha^2 + \beta^2)r^2 - p}{u},$$

so that

$$(10) \quad \{(\alpha^2 + \beta^2)r^2 - p\} \{u^2 + 2r\beta pu + p^2\{(\alpha^2 + \beta^2)r^2 - p\}\} = 0.$$

As before we consider the quadratic function

$$u^2 + 2r\beta pu + p^2\{(\alpha^2 + \beta^2)r^2 - p\}$$

The discriminant is  $4p^2(p - \alpha^2 r^2)$  which cannot vanish, so that, as before, the first factor in (10) must be zero, and we have

$$(11) \quad (\alpha^2 + \beta^2)r^2 - p = 0$$

which is a contradiction since  $\alpha^2 + \beta^2 > 1$  and we are supposing that  $|r| > 1$ .

Therefore we cannot have  $|r| > 1$ ,  $K \neq 0$ , and  $L \neq 0$ . If  $|r| = 1$  we see that  $K = L = 0$  since  $|x - \alpha r| < p$  and  $|y - \beta r| < p$  in this case. If  $|r| > 1$  with  $K = L = 0$  we would have  $x = \alpha r$ ,  $y = \beta r$  and hence  $(x, y) > 1$ , whereas  $x$  and  $y$  are relatively prime. Finally it remains to consider the possibility of having  $|r| > 1$  with one of  $K$  and  $L$  zero, the other nonzero. This if we suppose that  $|r| > 1$ ,  $K = 0$ ,  $L \neq 0$ , we obtain (9) which, as we have seen, leads to a contradiction. On the other hand the supposition that  $|r| > 1$  with  $K \neq 0$ ,  $L = 0$  implies that (11) would hold with  $r^2 > 1$ .

We conclude that  $|r| = 1$ ,  $K = 0$  and  $L = 0$ . Hence  $x = \pm \alpha$ ,  $y = \pm \beta$  and  $\alpha^2 + \beta^2 = p$ .

In [1], Corollary 2, we observed that if  $p = x^2 + y^2$  then, in our notation,  $y$  is a quadratic residue of  $p$ . Collecting our results we have the

COROLLARY. Let  $p = x^2 + y^2$  where  $p$  is a prime of the form  $4n + 1$  with  $x$  and  $y$  given by (3) and (4). Then  $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$  and  $\left(\frac{y}{p}\right) = 1$ .

## 5. CONCLUSION

We saw that  $x = \pm \alpha$ ,  $y = \pm \beta$ . When  $p = 13$  we have  $y = -3$ ,  $\beta = -3$ ; when  $p = 29$ ,  $y = -5$ ,  $\beta = 5$ , and when  $p = 41$ ,  $y = 5$ ,  $\beta = 5$ . Hence the sign of  $y$ , determined by the approximants to a continued fraction depends on the integer  $m$ , the number of terms in the finite segment of (2) which is used, can agree with that of  $\beta$  or be opposite that of  $\beta$ . The same applies to  $x$  and  $\alpha$ . In [1], Theorem 1, we gave a construction which always gives positive values for  $x$  and  $y$ . Other various constructions, as we have seen, do not have this property.

Finally we comment on the numbers  $\frac{(2n)!}{2(n!)^2}$  which we denote by  $a_n$  for  $n = 1, 2, 3, \dots$

The members of the sequence  $\{a_n\}$  are related to the numbers  $b_{n+1} = \frac{(2n)!}{(n+1)!n!}$ ,  $n = 0, 1, 2, \dots$ , which, as mentioned by Becker [2], have a variety of applications. Birkhoff [3] pointed out that  $b_n$  is an integer for every positive integer  $n$ , and noted the recurrence relation  $b_n = \sum_{i=1}^{n-1} b_i b_{n-i}$ ; a relation which was also obtained by Wedderburn [10].

The results of this note depend on the fact that  $a_n$  is an integer, at least when  $p = 4n + 1$  is a prime. Although it is known that  $a_n$  is an integer for every positive integer  $n$ , we can see that this also follows readily from [3]. For we have  $2a_n = (n+1)b_{n+1}$ . If  $n$  is even, it follows that  $b_{n+1}$  is even since  $(2, n+1) = 1$ . Therefore  $a_n = (n+1) \frac{b_{n+1}}{2}$  is an integer. If  $n$  is odd then  $2 \mid (n+1)$  and in this case also  $a_n = \frac{n+1}{2} b_{n+1}$  is an integer. A list of values for  $a_n$  can be obtained from the second column of a table in [2], page 699, headed  $N_n$ , by multiplying the  $(n+1)$ st member by  $\frac{n+1}{2}$ .

#### REFERENCES

- [1] BARNES, C. W. The Representation of Primes of the Form  $4n + 1$  as the Sum of Two Squares, *L'Enseignement Mathématique*, 18 (1972), pp. 289-299.
- [2] BECKER, H. W. Discussion of Problem 4277, *American Mathematical Monthly*, 56 (1949), pp. 697-699.
- [3] BIRKHOFF, Garrett. Problem 3674, *American Mathematical Monthly*, 42 (1935), pp. 518-521.
- [4] CAUCHY, A. *Sur les Formes quadratiques de certaines Puissances des Nombres Premiers ou du quadruple de ces Puissances*, Œuvres complètes 1<sup>re</sup> série, tome 3, pp. 390-437.
- [5] DAVENPORT, H. *The Higher Arithmetic*, Hutchinson's University Library, London, 1952.
- [6] GAUSS, C. F. Werke Bd. 2, S. 90-91.
- [7] JACOBSTHAL, Ernst. Über die Darstellung der Primzahlen der Form  $4n + 1$  als Summe Zweier Quadrate, *Journal für die Reine und Angewandte Mathematik*, Band 132, (1907), pp. 238-245.
- [8] LEGENDRE, A. M. *Théorie des Nombres*. Troisième édition, Paris, 1830.
- [9] PERRON, Oskar. *Die Lehre von den Kettenbrüchen*. Chelsea, New York, 1951.
- [10] WEDDERBURN, J. H. M. The Functional Equation  $g(x^2) = 2\alpha x + [g(x)]^2$ , *Annals of Mathematics*, (2), 24, (1922-1923), pp. 121-140.

(Reçu le 1<sup>er</sup> mai 1973)

C. W. Barnes  
 Department of Mathematics  
 University of Mississippi  
 Mississippi, 38677, USA



**Vide-leer-empty**