ON TRANSLATIVE SUBDIVISIONS OF CONVEX DOMAINS

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In euclidean n-space R^n let K be a convex body (compact convex subset of R^n with interior points). Let $\mathbf{S} = \{ S_1, S_2, \dots S_m \}$ be a finite collection of at least two closed subsets of K such that each S_i can be obtained from any S_j by a translation. Then, **S** will be called a *translative subdivision* of K if

(1) $$S_1 \cup S_2 \cup \dots \cup S_m = K ,$$

and if for $i \neq j$

$$\text{int } S_i \cap \text{int } S_j = \varnothing .$$

Under the assumption that the sets S_i of a translative subdivision of a convex body K are also convex it can be shown that K and the sets S_i must be cylinders (for $n=2$ parallelograms). Also, the possible arrangements of the sets S_i can be completely described (see [2]). Related to this result is the question whether there exist a convex body K and a translative subdivision $\{ S_1, S_2, \dots S_m \}$ of K with sets S_i that are not convex. If no assumptions concerning the regularity or connectivity of the sets S_i are made, there are trivial examples of convex bodies (e.g. cubes) which permit such non-convex subdivisions. To obtain a meaningful problem let us call a subset M of R^n *strongly connected* if any two of its points can be connected in the interior of M by a Jordan arc; that means, if $x \in M$, $y \in M$, $x \neq y$ there exists a Jordan arc γ with x and y as endpoints and such that every point of γ which is different from x and y is contained in the interior of M. Using this definition, the question can be raised whether there exists a convex body with a translative subdivision that consist of strongly connected non-convex sets. For $n = 1$ the situation is completely trivial. For $n \geqq 3$ this problem has not yet been solved. In the present paper the case $n = 2$ is settled by the following theorem which will be proved with the aid of the Jordan curve theorem. As a convenient abbreviation a two-dimensional convex body will be called a convex domain.

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THEOREM. *If a translative subdivision of a convex domain consists of strongly connected compact sets, then these sets are necessarily convex (and therefore parallelograms).*

PROOF: Let K be a given convex domain and let us assume that K has a translative subdivision $\{ S_1, S_2, \ldots S_m \}$ with strongly connected non-convex sets S_i. As a notational simplification, the set S_1 will often be denoted by S. Now there are two possibilities. Either the boundary of the convex hull of S is contained in S or this is not the case.

I. Assume that

$$\text{bdr conv } S \subset S \,, \tag{3}$$

where conv S denotes the convex hull of S. Since S is not convex there is a point p with $p \in \text{conv } S$ and

$$p \notin S \,. \tag{4}$$

Because of conv S = bdr conv $S \cup$ int conv S and (3) this implies

$$p \in \text{int conv } S \,. \tag{5}$$

From the convexity of K and $S \subset K$ it follows that conv $S \subset K$ and therefore

$$p \in K \,. \tag{6}$$

The relations (1), (4), and (6) imply

$$p \in S_j = S + t \tag{7}$$

for some $j \neq 1$ and a translation vector $t \neq 0$. The set $S + t$ is not contained in conv S (for this would imply (conv S) + t = conv $(S+t) \subset$ conv S which is clearly impossible since a convex domain cannot contain a translate of itself). Hence, there is a point q with

$$q \notin \text{conv } S \,, \tag{8}$$

$$q \in S + t \,. \tag{9}$$

(7) and (9) show that p and q can be connected in the interior of $S + t$ by some Jordan arc γ. From (5) and (8) one obtains that γ has a point, say x, in common with bdr conv S. Because of the assumption (3) it is clear that

$$x \in S \,. \tag{10}$$

On the other hand, (5) and (8) show that $x \neq p$, $x \neq q$ and therefore

(11) $$x \in \operatorname{int}(S+t) = \operatorname{int} S_j .$$

Because of (10) and the strong connectivity of S there are interior points of S in any neighborhood of x. This together with (11) shows that for some $j \neq 1$

$$\operatorname{int} S_1 \cap \operatorname{int} S_j \neq \varnothing$$

in contradiction to (2).

II. Assume that bdr conv $S \not\subset S$. This means that there exists a point g with $g \in$ bdr conv S and

(12) $$g \notin S .$$

By a well-known version of the theorem of Carathéodory on the convex hull of connected sets (see Bonnesen-Fenchel [1], p. 9) there is a closed line segment $\sigma = [s_1, s_2]$ with $s_1 \in S$, $s_2 \in S$ and g in its (relative) interior. If L denotes a support line for conv S which contains g, it is obvious that

(13) $$\sigma \subset L .$$

Let H be the halfplane which is bounded by L and contains conv S, and let H_i be defined by $H_i = H + t_i$ where t_i is the translation vector determined by $S_i = S + t_i$. Then, the union of all the halfplanes H_i is again one of these halfplanes, say H_k. Since H_k contains every S_i it follows that the line $L_k = L + t_k$ is a support line of K. By a proper assignment of the subscripts it can be achieved that $k = 1$ and therefore $L_k = L$. Hence, there is no loss in generality by assuming that the line L which contains σ is a support line of K. This implies in particular that

(14) $$\sigma \subset \operatorname{bdr} K .$$

Because of the strong connectivity of S it is possible to connect the points s_1 and s_2 in the interior of S by some Jordan arc τ. Since (13) implies that σ contains no interior points of S the arcs σ and τ have only the points s_1 and s_2 in common. Let λ be the closed Jordan curve composed of σ and τ. Then, the Jordan curve theorem shows that the complement of λ (with respect to R^2) consists of two open connected sets which have the same boundary, namely λ. Further, one of these regions, say J, is bounded and the other is unbounded.

From the inclusions $J \subset (J \cup \lambda) \subset \text{conv}\,(J \cup \lambda) = \text{conv}\,\lambda$ and $\lambda = (\sigma \cup \tau) \subset ((\text{conv}\,S) \cup S) \subset \text{conv}\,S \subset K$ it follows immediately that

$$(15) \qquad J \subset \text{conv}\,S$$

and

$$(16) \qquad J \subset K\,.$$

Because of (12) and the compactness of S it is obvious that the point g has positive distance from S. Using the fact that $g \in \lambda = \text{bdr}\,J$ one can find a point q in J which is so close to g that $q \notin S$. This, together with (1) and (16) shows that q is contained in some $S_h \neq S$. Actually, one may assume that q is in the interior of S_h. If necessary this can be achieved by a sufficiently small change in the selection of q without disturbing the relations $q \in J$, $q \notin S$. On the other hand, there is a point p with $p \in \text{int}\,S_h$ and $p \notin J$. If such a point would not exist one had $\text{int}\,S_h \subset J$. But then (15) shows that $\text{int}\,S_h \subset \text{conv}\,S$ and by taking the closure and the convex hull one would obtain $\text{conv}\,S_h = (\text{conv}\,S) + t_h \subset \text{conv}\,S$ with $t_h \neq 0$ and this is certainly impossible. Note that the closure of the interior of S_h is S_h since the strong connectivity implies that there are interior points in any neighborhood of a boundary point.

Hence, it has been found that there are points p, q with the following properties:

$$(17) \qquad p \in \text{int}\,S_h\,, \quad q \in \text{int}\,S_h\,,$$

$$(18) \qquad p \notin J\,, \quad q \in J\,.$$

Let κ be a Jordan arc which connects p and q in the interior of S_h. Because of (17) the endpoints of κ are also in $\text{int}\,S_h$ and therefore in $\text{int}\,K$. This fact, if compared with (14), shows that κ and σ are disjoint. On the other hand, it follows from (18) that κ must contain a point of $\text{bdr}\,J = \lambda = \tau \cup \sigma$. Writing $\tau' = \tau - \{\, s_1, s_2 \,\}$ it has therefore been shown that κ and τ' have a point, say x, in common. But this implies that $x \in \kappa \subset \text{int}\,S_h$ and $x \in \tau' \subset \text{int}\,S_1$, which contradicts the assumption (2).

REFERENCES

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