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A SIMPLE PROOF OF THE MAIN THEOREM OF ELIMINATION THEORY IN ALGEBRAIC GEOMETRY

by P. Cartier and J. Tate

Summary

The purpose of this note is to provide a simple proof (which we believe to be new) for the weak zero theorem in the case of homogeneous polynomials. From this theorem and Nakayama's lemma, we deduce easily the main theorem of elimination theory. Our version of elimination theory is given in very general terms allowing a straightforward translation into the language of schemes. Our proofs are highly non constructive—the price we pay for simplicity and elegance.

We thank N. Bourbaki for numerous lively discussions about the subject matter of this note.

1. Hilbert's zero theorem: a particular case

We denote by k a field and K an algebraically closed extension of k. The statement of Hilbert's zero theorem, in its weak form for homogeneous polynomials, reads as follows:

Theorem A. *Let n be a nonnegative integer and J an ideal in the polynomial ring $k[X_0, X_1, ..., X_n]$ generated by homogeneous polynomials. One has the following dichotomy:*

a) *Either there exists a nonnegative integer d_0 such that J contains every homogeneous polynomial of degree $d \geqslant d_0$;*

b) *or there exists a nonzero vector $\xi = (\xi_0, \xi_1, ..., \xi_n)$ with coordinates from K such that $P(\xi) = 0$ holds for any polynomial P in J.*

We begin by reformulating the previous theorem. It is immediate that properties a) and b) are mutually exclusive. For any nonnegative integer d, let S_d be the vector space (over k) consisting of the polynomials in the ring $S = k[X_0, X_1, ..., X_n]$ which are homogeneous of degree d. Then

$S = \bigoplus_{d \geqq 0} S_d$, and for the multiplication one gets $S_d \,.\, S_e \subset S_{d+e}$. Otherwise stated, *S is a graded algebra over the field k*. Since J is generated by homogeneous polynomials, it is a *graded ideal*, namely $J = \bigoplus_{d \geqq 0} (J \cap S_d)$. The factor algebra $R = S/J$ is therefore graded with $R_d = S_d/(J \cap S_d)$ for any nonnegative integer d. It enjoys the following properties:

(i) *As a ring,* R *is generated by* $R_0 \cup R_1$.

(ii) *For any nonnegative integer* d, *the vector space* R_d *is finite-dimensional over* k.

(iii) $R_0 = k$.

Denote by $x_0, x_1, ..., x_n$ respectively the cosets of $X_0, X_1, ..., X_n$ modulo J. Let φ be any k-linear ring homomorphism from R into K, and put $\xi_0 = \varphi(x_0), ..., \xi_n = \varphi(x_n)$. It is clear that the vector $\xi = (\xi_0, \xi_1, ..., \xi_n)$ is a common zero of the polynomials in J. Conversely, for any such common zero, there exists a unique k-linear ring homomorphism $\varphi : R \to K$ such that $\xi_0 = \varphi(x_0), ..., \xi_n = \varphi(x_n)$. The vector ξ is equal to zero if and only if φ maps $R_1 = kx_0 + ... + kx_n$ onto 0, that is if and only if the kernel of φ is equal to the ideal $R^+ = \bigoplus_{d \geqq 0} R_d$ in R.

Theorem A is therefore equivalent to the following.

THEOREM B. *Let* R *be a graded commutative algebra over* k, *satisfying hypotheses* (i), (ii) *and* (iii) *above. One has the following dichotomy:*

a) *Either there exists a non-negative integer* d_0 *such that* $R_d = 0$ *for* $d \geqslant d_0$;

b) *or for every nonnegative integer* d, *one has* $R_d \neq 0$ *and there exists a k-linear ring homomorphism* $\varphi : R \to K$ *whose kernel is different from* $R^+ = \bigoplus_{d \geqq 1} R_d$.

Notice that R is a finite-dimensional vector space in case a), infinite-dimensional in case b).

2. Proof of Hilbert's zero theorem

We proceed to the proof of theorem B.

By property (i) above, one gets $R_1 \,.\, R_d = R_{d+1}$ hence $R_d = 0$ implies $R_{d+1} = 0$. Hence either R_d is 0 for all sufficiently large d's, or $R_d \neq 0$