Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COINCIDENCE-FIXED-POINT INDEX
Autor: Dold, Albrecht
DOI: https://doi.org/10.5169/seals-49689

A COINCIDENCE-FIXED-POINT INDEX [1]

by Albrecht Dold

B. Eckmann anlässlich seines 60. Geburtstages gewidmet

Introduction

The fixed point set of a map $\varphi: X \to X$ is, generically, a discrete set; if it is compact its (weighted) cardinality is measured by the Hopf-index $I(\varphi) \in \mathbf{Z}$. The coincidence set K of a pair of maps $(\varphi, p): X \rightrightarrows Y$ is not discrete; its generic dimension is $\dim K = \dim X - \dim Y$. If K is compact it can sometimes (compare 3.8) be measured by a cohomology invariant κ, but even then κ is difficult to deal with. This might explain why most studies on coincidence questions make additional assumptions on (φ, p), or use auxiliary data. For instance, if one of the maps, say p, admits a section of sorts σ then the fixed points of $\sigma\varphi$ are in K so that fixed point methods give coincidence results. Usually σ is not a genuine section; for instance, if p is a Vietoris map then one uses $(p^*)^{-1}$, on the cohomology level (cf. 3.7).

The idea of the present lecture is to let fixed point transfers in the sense of [2] play the role of σ; we have to assume, therefore, that p is ENR_Y which means (roughly speaking; cf. [2]) that p has sufficiently many local sections. Actually, our procedure for counting fixed points of $\sigma\varphi$ (cf. §1) is much more elementary than [2] and doesn't really use transfers. Only when we express the number of fixed points of $\sigma\varphi$ as a Lefschetz trace in theorem 2.1, transfers t become essential. If one imposes further (rather restrictive) assumptions on p then t can be eliminated again (from the theorem; it is still used in the proof), as shown in prop. 3.5. — The last section of the paper discusses applications (3.1-3.6) and problems (3.7, 3.8).

---

[1]) Presented at the Colloquium on Topology and Algebra, April 1977, Zurich.

§ 1. The coincidence-fixed-point (c.f.p.) index

(1.1) Let $p: E \to B$ denote a euclidean neighborhood retract over B (abbrev. ENR_B), where B, and hence E, is an ENR. Altogether this means that $p: E \to B$ embeds as a neighborhood retract into the projection $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$, for some m, n. We refer the reader to [2], §1, for the precise definitions but remark that every smooth submersion and every fibration (with base and total space ENR) qualifies for $p: E \to B$.

We consider continuous maps $g: D_g \to E$, $\varphi: D_\varphi \to B$, where D_g, D_φ are open subsets of E, and $pg = p \mid D_g$ (i.e., g is fibre-preserving). We let $\mathrm{Fix}\,(g) = \{ x \in D_g \mid gx = x \}$ and $\mathrm{Coinc}\,(\varphi, p) = \{ x \in D_\varphi \mid \varphi x = px \}$, and we *assume that* $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ *is compact*. Under these circumstances we shall define an integer $J(g, \varphi) \in \mathbf{Z}$ which is akin to the Hopf fixed-point index. It "counts" the points in $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ in a weighted and homotopy-invariant fashion. It *is* the Hopf index of g resp. φ if B is a single point resp. p is the identity map of B.

(1.2) By definition [2], 1.1 of an ENR_B, we have that E is a fibre-preserving neighborhood retract of some $\mathbf{R}^n \times B$. In fact, for the present purpose we can use any product $Y \times B$, i.e. we'll use mappings $E \xrightarrow{i} V \xrightarrow{r} E$ such that $V \subset Y \times B$ is open, $ri = id$, and i, r are maps over B. In formulas,

$$ix = (i'x, px)\,, \quad \text{where } i': E \to Y\,, \tag{1.3}$$

$$pr\,(y, b) = b\,, \quad \text{for } (y, b) \in V\,, \tag{1.4}$$

$$r\,(i'x, px) = x\,, \quad \text{for } x \in E\,. \tag{1.5}$$

Consider the following sequence of maps

$$D_g \cap D_\varphi \xrightarrow{(g,\varphi)} E \times B \xrightarrow{i' \times id} Y \times B \supset V \xrightarrow{r} E\,. \tag{1.6}$$

Its composite $[g, \varphi]$ is defined in $D_V = (i'g, \varphi)^{-1}\,V$ which is an open subset of $(D_g \cap D_\varphi)$, and hence of E. Thus

$$[g, \varphi]: D_V \to E\,, \quad [g, \varphi]\,(x) = r\,(i'gx, \varphi x)\,. \tag{1.7}$$

If $x \in D_g \cap \mathrm{Coinc}\,(\varphi, p)$ then

$$(i'g, \varphi)\,x = (i'gx, px) = (i'gx, pgx) = igx \in V\,,$$

hence $[g, \varphi]\,x$ is defined and equals $rigx = gx$. It follows that $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p) \subset \mathrm{Fix}\,[g, \varphi] = \{ x \in D_V \mid [g, \varphi]\,x = x \}$. Converseley, x

$= [g, \varphi]\, x$ implies $px = p\,[g, \varphi]\, x = pr\,(i'gx, \varphi x) = \varphi x$, hence $x \in \mathrm{Coinc}\,(\varphi, p)$, and $gx = [g, \varphi]\, x = x$. Altogether

$$\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p) = \mathrm{Fix}\,[g, \varphi]\,. \tag{1.8}$$

In particular, $[g, \varphi]\colon D_V \to E$ has a compact fixed-point set, and we can assign to it its Hopf-index $I\,[g, \varphi] \in \mathbf{Z}$ — for instance as in [1], VII,5.10. Furthermore,

(1.9) PROPOSITION AND DEFINITION. *The Hopf-index* $I\,[g, \varphi] \in \mathbf{Z}$ *depends only on* (g, φ), *not on the choice of the neighborhood retraction* i, r. We denote this integer by $J\,(g, \varphi)$, and call it the *c.f.p.-index* of (g, φ); thus $J\,(g, \varphi) = I\,[g, \varphi]$.

Proof. Because the range B of the maps φ, p is ENR, these two maps are homotopic in a neighborhood of $\mathrm{Coinc}\,(\varphi, p)$. In fact (cf. [1], IV,8.6), there is an open neighborhood U of $\mathrm{Coinc}\,(\varphi, p)$ in D_φ, and a deformation $\vartheta_t\colon U \to B$, $0 \leqq t \leqq 1$, such that

$$\vartheta_0 = p \mid U,\ \vartheta_1 = \varphi \mid U,\ \vartheta_t x = px \text{ for } x \in \mathrm{Coinc}\,(\varphi, p) \text{ and all } t. \tag{1.10}$$

Consider then two neighborhood retractions

$$E \xrightarrow{i} V \xrightarrow{r} E\,,\quad V \subset Y \times B\,;\ ix = (i'x, px)\,,$$
$$E \xrightarrow{j} W \xrightarrow{s} E\,,\quad W \subset Z \times B\,;\ jx = (j'x, px)\,,$$

as above, and the corresponding maps $[g, \varphi]_1$, $[g, \varphi]_2$ as defined by 1.6. We have to show $I\,([g, \varphi]_1) = I\,([g, \varphi]_2)$. In order to do so we can (cf. [1], VII,5.11) restrict attention to an arbitrary open neighborhood N of $\mathrm{Fix}\,([g, \varphi]_i) = \mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$. And we shall show that $[g, \varphi]_i \mid N$ are homotopic $(i=1, 2)$ without moving the fixed point set, provided N is sufficiently small. The homotopy is given by the formula

$$\theta_t x = s\big(j'r\,(i'gx, \vartheta_t x)\,,\ \varphi x\big)\,. \tag{1.11}$$

This is defined for (x, t) such that $x \in D_g \cap U$, $v = (i'gx, \vartheta_t x) \in V$, and $w = (j'rv, \varphi x) \in W$; the set of all such (x, t) is an open subset D_θ of $E \times [0, 1]$. If $x \in \mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ then

$$v = (i'gx, \vartheta_t x) = (i'x, px) = ix \in V\,, \text{ and } rv = x\,,$$

hence

$$w = (j'rv, \varphi x) = (j'x, px) = jx \in W\,, \text{ and } \theta_t x = sw = x\,.$$

Therefore, $(\mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi, p)) \times [0, 1] \subset D_\theta$, and $(\mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi, p)) \subset \mathrm{Fix}(\theta_t)$ for all t. It follows that

$$N = \{x \in E \mid (x, t) \in D_\theta \text{ for all } t\}$$

is an open neighborhood of $\mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi, p)$ in which the deformation θ is defined (by 1.11).

Suppose now $x \in N$ is a fixed point of θ_t, thus $x = s(j'r(i'gx, \vartheta_t x), \varphi x)$. Apply p, using 1.4 for s, and get $px = \varphi x =$, hence $\vartheta_t x = px$ by 1.10, hence $r(i'gx, \vartheta_t x) = r(i'gx, px) = r(i'gx, pgx) = rigx = gx$, hence $x = \theta_t x = s(j'gx, px) = s(j'gx, pgx) = sjgx = gx$; altogether, $x \in \mathrm{Coinc}(\varphi, p) \cap \mathrm{Fix}(g)$. It follows that the fixed point set $\mathrm{Fix}(\theta_t) = \mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi,p)$ for all t. In particular, $\cup_{t\in[0,1]} \mathrm{Fix}(\theta_t)$ is compact, hence (cf. [1].VII,5.15) all θ_t have the same Hopf-index $I(\theta_t)$. But $r(i'gx, \vartheta_0 x) = r(i'gx, px) = r(i'gx, pgx) = gx$, hence $\theta_0 x = s(j'gx, \varphi x) = [g, \varphi]_2 x$. To calculate θ_1 we first remark that $p[g, \varphi]_1 x = \varphi x$, by 1.7 and 1.4; also $r(i'gx, \vartheta_1 x) = r(i'gx, \varphi x) = [g, \varphi]_1 x$, hence $\theta_1 x = s(j'[g, \varphi]_1 x, p[g, \varphi]_1 x = sj[g,\varphi]_1 x = [g, \varphi]_1 x$. □

(1.12) *The product case* $E = F \times B$, $p =$ *projection*. In this case $g\colon D_g \to F \times B$ has the form $g(y, b) = (\gamma(y, b), b)$ with $\gamma\colon D_g \to F$. The two maps (γ, φ) combine to a map $(\gamma, \varphi)\colon D \to F \times B$, where $D(=D_g \cap D_\varphi)$ is an open subset of $F \times B$, and $\mathrm{Fix}(\gamma, \varphi) = \mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi, p)$. In order to obtain the c.f.p.-index $J(g, \varphi)$ one can use $Y = F$ and the neighborhood retraction $i = r =$ identity-map of $Y \times B$. The definition 1.9 then shows that

$$J(g, \varphi) = I(\gamma, \varphi);$$

i.e. *in the product case the c.f.p.-index of* (g, φ) *is simply the Hopf-index of* $(y, b) \mapsto (\gamma(y, b), \varphi(y, b))$.

The procedure 1.6-1.9 in the general case, on the other hand, can be considered as a reduction to the product case.

(1.13) *General properties of* $J(g, \varphi)$ follow from corresponding properties of the Hopf-index. For instance, $J(g, \varphi)$ is additive with respect to topological-sum decompositions of $\mathrm{Fix}(g) \cap \mathrm{Coinc}(g, \varphi)$, it is invariant under deformations such that $\bigcup_{0 \leq t \leq 1} \mathrm{Fix}(g_t) \cap \mathrm{Coinc}(\varphi_t, p)$ is compact, it depends only on the germ of (g, φ) around $\mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi, p)$ — in particular, $J(g, \varphi) = 0$ if $\mathrm{Fix}(g) \cap \mathrm{Coinc}(\varphi, p) = \varnothing$, etc. These details are left to the reader. Lefschetz-trace formulas for $J(g, \varphi)$ can be found in 2.1 and 3.5.

§ 2. The Lefschetz trace formula for the c.f.p. index

This reduces to the classical Lefschetz-Hopf theorem if B = a point, or if $p: E = B$. Our assumptions in 2.1 are a little more restrictive than necessary, in order to facilitate the proof; a slight generalization is indicated in 2.8.

(2.1) Theorem. *Let* $p: E \to B$ *be an* ENR_B, *where* B *is a* compact *ENR. Let* $g: D_g \to E$, $\varphi: D_\varphi \to B$ *denote maps as in 1.1 such that* Fix (g) is compact, and $D_\varphi \supset$ Fix (g). *Then the c.f.p. index of* (g, φ) *agrees with the Lefschetz trace of the composite* $\check{h}B \xrightarrow{\varphi} \check{h}\,\text{Fix}(g) \xrightarrow{t} \check{h}B$, *or* $hB \xrightarrow{\varphi^*} hD \xrightarrow{t^D} hB$, *where* $t = t_g$ *is the fixed-point transfer* (cf. [2], § 3), D *is any neighborhood of* Fix (g) *in* D_φ, h *is singular and* $\check{h}$ *is* $\check{C}$*ech-cohomology with coefficients in* $\mathbf{Z}$ *or* $\mathbf{Q}$. In formulas,

$$J(g, \varphi) = tr(t_g \circ \check{\varphi}) = tr(t_g^D \circ \varphi^*) . \tag{2.2}$$

Proof. Using a vertical neighborhood retraction we can assume that $E = \mathbf{R}^n \times B$; this is, in fact, what the definition 1.6-1.9 shows (if $Y = \mathbf{R}^n$). Then $g(y, b) = (\gamma(y, b), b)$, where $\gamma: D_g \to \mathbf{R}^n$, and $J(g, \varphi) = I(\gamma, \varphi)$ as explained in 1.12. Furthermore, since B is ENR, we have $\iota: B \subset U \subset \mathbf{R}^m$ and a retraction $\rho: U \to B$, where U is open in $\mathbf{R}^m$. We can then extend φ, γ, g to maps $\tilde{\varphi}, \tilde{\gamma}, \tilde{g}$ of open subsets of $\mathbf{R}^n \times U \subset \mathbf{R}^n \times \mathbf{R}^m$ by composing with $id \times \rho: \mathbf{R}^n \times U \to \mathbf{R}^n \times B$. The fixed points of (γ, φ), $(\tilde{\gamma}, \tilde{\varphi})$ (and their index) are the same, by commutativity [1], VIII, 5.16 — since $(\tilde{\gamma}, \tilde{\varphi}) = (id \times \iota)(\gamma, \varphi)(id \times \rho)$. Altogether (omitting the $\tilde{}$), we can assume that φ, γ, g are defined in open subsets D_φ, $D_\gamma = D_g$ of $\mathbf{R}^n \times U$, $\varphi: D_\varphi \to B \subset U$, $\gamma: D_g \to \mathbf{R}^n$, $D_\varphi \supset$ Fix (g), Fix (g) is (no longer compact but) proper over U; in particular, $K =$ Fix $(g) \cap (p^{-1}B)$ is compact.

We now argue in a similar (although simpler) fashion as on p. 241 of [2]. We consider the following diagram (explanations below).

(2.3)

$$
\begin{array}{ccccccc}
\mathbf{R}^n_o \times \mathbf{R}^m_o & \overset{\alpha}{\dashrightarrow} & (X, X-K) & & \xrightarrow{(q\text{-}\gamma,\ p\text{-}\varphi)} & & \mathbf{R}^n_o \times \mathbf{R}^m_o \\
\Big\downarrow{\scriptstyle id \times j} & & \Big\| & & & & \Big\uparrow{\scriptstyle id \times d} \\
\mathbf{R}^n_o \times (U, U-B) & \overset{\beta}{\dashrightarrow} & \big(X, (X-\mathrm{Fix}(g)) \cup (X-X_B)\big) & \xrightarrow{(q\text{-}\gamma,\ id)} & \mathbf{R}^n_o \times (X, X-X_B) & \xrightarrow{id \times (p,\ \varphi)} & \mathbf{R}^n_o \times (U, U-B) \times B
\end{array}
$$

$$t_g : \mathbf{R}^n_o \times (U, U-B) \to \mathbf{R}^n_o \times (X, X-X_B)$$

Here, $\mathbf{R}_o^n = (\mathbf{R}^n, \mathbf{R}^n - 0)$, X is an open neighborhood of $\mathrm{Fix}(g)$ in which γ and φ are defined, $K = \mathrm{Fix}(g) \cap (p^{-1}B)$, $X_B = X \cap (p^{-1}B)$, $q\colon X \subset \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is the projection, $d(u, b) = u - b$. The dotted arrows stand for sequences of inclusion maps (as in [2], 3.3); some of these go the wrong way but then they are homotopy equivalences or excisions, inducing isomorphisms in cohomology. For instance, α stands for

$$\mathbf{R}_o^n \times \mathbf{R}_o^m = \mathbf{R}_o^{n+m} \sim (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - C) \hookrightarrow (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - K) \xleftarrow{EXC} (X, X\text{-}K)$$

where C is a ball around 0, containing K. Similarly for j on the left. β is a relative version (compare [2], 3.7), namely

$$\begin{array}{l} \mathbf{R}_o^n \times (U, U - B) \sim (\mathbf{R}^n, \mathbf{R}^n - C') \times (U, U - B) \hookrightarrow \\ \big(\mathbf{R}^n \times U,\ (\mathbf{R}^n \times U - \mathrm{Fix}(g)) \cup (\mathbf{R}^n \times (U - B)\big) \xleftarrow{EXC} \\ \big(X, (X - \mathrm{Fix}(g)) \cup (X - X_B)\big), \end{array}$$

where C' is a ball around $0 \in \mathbf{R}^n$ such that $K \subset (C' \times B)$. The lower t_g will be explained later.

The reader might want to follow the track of an element across the diagram 2.3; it looks as follows

$$\begin{array}{ccccc} (y, b) \mapsto (y, b) & \longmapsto & & & (y - \gamma(y, b),\ b - \varphi(y, b)) \\ \downarrow \quad\ \| & & & & \uparrow \\ (y, b) \mapsto (y, b) & \longmapsto & (y - \gamma(y, b),\ y, b) & \longmapsto & (y - \gamma(y, b), b, \varphi(y, b)). \end{array}$$

We now apply cohomology $h = H^*(-; \mathbf{Q})$ to the diagram 2.3. Let $s^n \in h^n \mathbf{R}_o^n$ the canonical generator. Then $s^n \times s^m$ generates $h^{n+m}(\mathbf{R}_o^n \times \mathbf{R}_o^m)$, and its image along the top row of 2.3 is $I(g, \gamma)\, s^n \times s^m = J(g, \varphi)\, s^n \times s^m$, by definitions [1], VII, 5.2, and 1.9 above.

The left part of the lower row (which is marked t_g) induces the relative transfer (or trace) homomorphism $t_g\colon h(X, X - X_B) \to h(U, U - B)$, as defined in [2], 3.6-8. In formulas,

(2.4) $$s^n \times \xi \quad s^n \mapsto \times\, t_g^{X, Z}(\xi), \quad Z = X - X_B.$$

Actually, [2], 3.8 is a little more general: it maps $h(X, X - X_B)$ into $h(U, \tilde{U})$, where $\tilde{U} \supset (U - B)$; we've composed [2], 3.8 with $h(U, \tilde{U}) \to h(U, U - B)$.

Using the Künneth-formula we can write

(2.5) $$d^* s^m = \sum_\nu \alpha_\nu \times \beta_\nu\,, \text{ with } \alpha_\nu \in h(U, U-B)\,,\ \beta_\nu \in hB\,.$$

Following $\alpha_\nu \times \beta_\nu$ along the lower row of (2.3) gives

(2.6) $$\alpha_\nu \times \beta_\nu \mapsto t_g(p^*\alpha_\nu \smile \varphi^*\beta_\nu) = \alpha_\nu \smile (t_g\varphi^*\beta_\nu)\,,$$

the latter because t_g is a homomorphism of modules over $h(U, U-B)$, by the relative version of [2], 3.20.

If we define $\kappa\colon h(U, U-B) \to \mathbf{Q}$ by $j^*(u) = \kappa(u)\, s^m$ (this corresponds to γ on p. 233, line 3^- of [2]), then $s^n \times \alpha_\nu \times \beta_\nu$ has image $\kappa(\alpha_\nu \smile t_g\varphi^*\beta_\nu)\, s^n \times s^m$ in the upper left corner of 2.3. On the other hand $\kappa(\alpha_\nu \smile t_g\varphi^*\beta_\nu)$ is the trace of the endomorphism

$$\xi \mapsto (-1)^{|\beta_\nu|}\beta_\nu\, \kappa(\alpha_\nu \smile t_g\varphi^*\xi)\,,\quad \xi \in hB\,,$$

by [2], 6.7. It follows, that the image of $d^*s^m = \sum_\nu s^n \times \alpha_\nu \times \beta_\nu$ in the upper left corner is $s^n \times s^m$-times the trace of

(2.7) $$\xi \mapsto \sum_\nu (-1)^{|\beta_\nu|}\beta_\nu\, \kappa(\alpha_\nu \smile t_g\varphi^*\xi)\,,\quad \xi \in hB\,,$$

and so $J(g, \varphi) =$ trace of 2.7.

It remains to show that 2.7 agrees with $\overset{B}{t}_g \overset{*}{\varphi}_B$, where we now add indices (B, or U) to indicate the range of t_g resp. the domaine of φ^*. This will follow from [2], 6.16 which asserts (in greater generality) that $\sum_\nu (-1)^{|\beta_\nu|}\beta_\nu\, \kappa(\alpha_\nu \smile \eta)$ $= \iota^*\eta$, for $\eta \in hU$ and $\iota^*\colon hU \to hB$. Taking $\eta = \overset{U}{t}_g \overset{*}{\varphi}_U \xi$ we see that 2.7 agrees with $\xi \mapsto \iota^* \overset{g}{t}_U \overset{*}{\varphi}_U \xi = \overset{B}{t}_g \overset{*}{\varphi}_B \xi$, the latter by naturality ([2], 3.12) of t_g applied to ι. □

(2.8) *Remark.* The assumption in 2.1 that B be compact can be weakened: It suffices that for some compact subset $R \subset B$ we have that $\operatorname{Fix}(g)_R = \operatorname{Fix}(g) \cap (p^{-1}R)$ *is compact, and*

$$\operatorname{im}(\varphi) \subset R\,,\ D_\varphi \supset \operatorname{Fix}(g)_R\,.$$

Then the composite $\check{h}R \xrightarrow{\check{\varphi}} \check{h}(\operatorname{Fix}(g)_R) \xrightarrow{t_g} \check{h}R$ *is defined, has finite rank, and has Lefschetz trace equal to* $J(g, \varphi)$.

Our proof of 2.1 can be adapted to this more general situation. *Or*, by arguments as in [2], 8.6, one can slightly increase R in B, and decrease D_φ, such that the increased R is a compact ENR, and over (the increased) R the assumptions of 2.1 are satisfied; then 2.1 will imply the more general result above.

§ 3. Applications, Problems.

(3.1) Whether and how the trace formula 2.1 can be used depends mainly on one's knowledge of the transfer t_g. For instance, one knows that

(i) $t_g p^* = I(g_b) =$ multiplication with the Hopf-index of $g_b: D_g \cap p^{-1}b \to p^{-1}b$ (in ordinary cohomology, B connected).

(ii) $t_g: hD_g \to hB$ is induced by a stable map of B^+ into D_g^+; in particular, it commutes with stable cohomology operations.

(iii) t_g is itself given by a trace-formula if $p: E \to B$ is a bundle with compact fibres which are totally non-cohomologous to zero.

We shall now illustrate (cf. 3.2, 3.3, 3.5) how these properties can be used.

(3.2) *Suppose* φ *is homotopic to* $\beta(p \mid D_\varphi)$, *for some* $\beta: B \to B$. Then $t_g\varphi^* = t_g p^* \beta^* = I(g_b)\beta^*$, provided B is connected (cf. [2], 4.8). Therefore

$$J(g, \varphi) = tr(t_g \varphi^*) = I(g_b)\, tr(\beta^*) = I(g_b)\, I(\beta)\,.$$

Geometrically, this result is very plausible: If $\varphi = \beta_p$ then Coinc (φ, p) consists of all fibres $D_\varphi \cap p^{-1}b$ with $b \in \mathrm{Fix}(\beta)$. The "number" of these fibres is $I(\beta)$, and in every fibre the "number" of fixed points of g equals $I(g_b)$. — As the geometry suggests, the result holds under more general assumptions and can be proved directly from § 1 (it doesn't seriously use 2.1).

As an illustration, the reader might look at the case where $p: E \to B$ is the tangent sphere-bundle of a compact Riemannian manifold B, and $\varphi = \varphi_t: E \to B$, $\varphi(x) = \exp(tx)$, for $t \in \mathbf{R}$. Clearly $\varphi \simeq \varphi_0 = p$, and Coinc $(\varphi, p) = \varnothing$ if $|t|$ is small enough, $t \neq 0$. Hence, $0 = J(g, \varphi) = I(g_b)\, I(id_B) = I(g_b)\, \chi(B)$, for all g. (For a direct proof of this result the reader should think of $\mathrm{Fix}(g) \subset E$ as a manifold such that $p \mid \mathrm{Fix}(g)$ has degree $I(g_b)$).

(3.3) The definition [2], 3.3-4 shows that t_g is a composite of geometric homomorphisms (induced by continuous maps) and suspension isomorphisms $h^j Y \cong h^{j+n}((\mathbf{R}^n, \mathbf{R}^n - 0) \times Y)$. Thus, t_g is induced by a stable map $B^+ \to D_g^+$ (in fact, by a stable shape map $B^+ \to \mathrm{Fix}(g)^+$); it commutes with stable cohomology operations, such as Steenrod's Sq^i or P^i. As a (rather weak) consequence of theorem 2.1 we obtain that *under the assumptions of 2.1 the c.f.p.-index* $J(g, \varphi)$ *is the Lefschetz trace of a homomorphism* $hB \to hB$ *which is induced by a stable map* $B^+ \to B^+$; *this homomorphism (namely* $t_g \varphi^*$*) satisfies* $1 \mapsto I(g_b) \cdot 1$.

For example, let $B = P_{2m}\mathbf{C}$ = complex projective $2m$-space, hence $H^*(B; R) = R[u]/(u^{2m+1})$, with $u \in H^2$, R any ring. If $R = \mathbf{Z}/2\mathbf{Z}$ then $Sq^2 u^{2i-1} = u^{2i}$; a stable map α must therefore satisfy $\alpha^* u^j = \lambda_j u^j$ with $\lambda_{2i-1} = \lambda_{2i}$. For integral coefficients $R = \mathbf{Z}$ this means $\lambda_{2i-1} \equiv \lambda_{2i} \bmod 2$. Therefore $tr(\alpha^*) \equiv \lambda_0 \bmod 2$. In our case $\alpha = t_g \varphi^*$ this says:

Under the assumptions of 2.1 and with $B = P_{2m}\mathbf{C}$ *the c.f.p.-index satisfies* $J(g, \varphi) \equiv I(g_b)$ *mod 2.* In particular, *if* $I(g_b)$ *is odd then* $J(g, \varphi) \neq 0$, *hence every* φ *has coincidence points with* p.

It is interesting to compare this result with [3], where the product case $E = Y \times P_{2m}\mathbf{C}$ is treated by different methods. It is shown there (compare also 1.12) that $J(g, \varphi)$, for globally defined (g, φ), is equal to $I(g_b)$ times an odd integer; in particular, $I(g_b) \neq 0 \Rightarrow J(g, \varphi) \neq 0$. One might wonder whether this extends to general bundles over $P_{2m}\mathbf{C}$, but the following example shows that it doesn't. Let $B = P_{2m}\mathbf{C}$, $E = B \times B - \tilde{\Delta}$ where $\tilde{\Delta}$ is an open tubular neighborhood of the diagonal, φ and p the two projections onto B, $g = id_E$. Then p is a bundle projection with compact fibre $\simeq P_{2m-1}\mathbf{C}$, $I(g_b) = \chi(\text{fibre}) = 2m \neq 0$, but $\mathrm{Coinc}(\varphi, p) = \varnothing$.

(3.4) If $p: E \to B$ is a fibration (where E and B are compact ENR, B connected) and if the fibre $Y = p^{-1}(b)$ is totally non-cohomologous to zero, i.e. $hE \to hY$ is epimorphic for $h = H^*(-; \mathbf{Q})$, then E is h-flat over B in the sense of [2], 6.9; in fact, hE has a Leray-Hirsch basis ([2], 6.8) over hB. In particular, $hE \cong hY \otimes hB$, as hB-modules (but not as rings, in general). In this case, [2], 6.18 expresses t_g in terms of Lefschetz traces over the ring hB. One can combine the two trace-formulas 2.1 and [2], 6.18, as follows.

(3.5) Proposition. *Let* $p: E \to B$ *a fibration between compact ENR-spaces* E, B *(*B *connected), and let* $\iota: Y \hookrightarrow E$ *the inclusion of the fibre. Assume* $hE = hY \otimes hB$ *as* hB*-modules, and such that* $\iota^*(y \otimes 1) = y$ *for* $y \in hY$, *where* $h = H^*(-; \mathbf{Q})$. *Then for every map* $\varphi: E \to B$ *and fibre-*

preserving map $g: E \to E\,(pg = p)$ *the c.f.p.-index* $J(g, \varphi)$ *equals the Lefschetz trace of*

$$hY \otimes hB \to hY \otimes hB\,, \; y \otimes z \mapsto g^* (y \otimes 1) \smile (\varphi^* z)\,.$$

Heuristically, this is found by pretending that the isomorphism $hE = hY \otimes hB$ comes from a product representation, and by comparing 2.1 with the discussion 1.12 of the product case. In order to actually prove it, we consider the following purely algebraic construction. For every $\alpha \in \mathrm{Hom}_{hB}(hE, hE) = \mathrm{Hom}_{\mathbf{Q}}(hY, hE)$ we define $\tau_\alpha \in \mathrm{Hom}_{hB}(hE, hB)$ by $\tau_\alpha(\xi) = tr\,(\tilde{\xi} \circ \alpha)$, where $\xi \in hE$ and $\tilde{\xi} \in \mathrm{Hom}_{hB}(hE, hE)$ is left translation with ξ, $\tilde{\xi}(x) = \xi \smile x$. For $\beta \in \mathrm{Hom}_{\mathbf{Q}}(hB, hE)$ and α as above, we define $\{\alpha, \beta\} \in \mathrm{Hom}_{\mathbf{Q}}(hE, hE)$ by $\{\alpha, \beta\}(y \otimes z) = \alpha(y \otimes 1)(\beta z)$. We assert,

$$tr\,\{\alpha, \beta\} = tr\,(\tau_\alpha \circ \beta)\,. \tag{3.6}$$

If we take $\alpha = g^*$ then $\tau_\alpha = t_g$, by [2], 6.18. If, moreover, $\beta = \varphi^*$ then 3.6 becomes 3.5, by 2.1. Thus, it remains to give a

Proof of 3.6. Let $\{y_i\}$ resp. $\{z_j\}$ denote bases of $hY = H^*(Y; \mathbf{Q})$ resp. $hB = H^*(B; \mathbf{Q})$. Since both sides of 3.6 are bilinear in (α, β) it suffices to consider the case where α and β vanish on all but one basic element y_i resp. z_j; thus, $\alpha(y_\mu) = 0$ for $\mu \neq i$, $\beta(z_\nu) = 0$ for $\nu \neq j$. Then

$$\{\alpha, \beta\}(y_i \otimes z_j) = (\alpha y_i) \smile (\beta z_j) = \lambda (y_i \otimes z_j) + \rho\,,$$

where $\lambda \in \mathbf{Q}$, and the remainder term ρ is irrelevant for the trace; hence, $tr\,\{\alpha, \beta\} = (-1)^{|y_i| + |z_j|}\lambda$, where $|\;|$ denotes dimension. Similarly, $(\widetilde{(\beta z_j)} \circ \alpha)(y_i) = (\beta z_j) \smile (\alpha y_i) = (-1)^{|y_i||z_j|}\, y_i \otimes (\lambda z_j) + \rho$, hence $(\tau_\alpha \circ \beta)(z_j) = tr\,(\widetilde{(\beta z_j)} \circ \alpha) = (-1)^{|y_i|}\lambda z_j + \rho'$ by [2], 6.6, hence $tr\,(\tau_\alpha \circ \beta) = (-1)^{|z_j|}(-1)^{|y_i|}\lambda$. □

(3.7) *Multivalued maps* $\beta: B \to B$ are usually given by, resp. related to pairs of ordinary maps $B \xleftarrow{p} E \xrightarrow{\varphi} B$ such that $\beta(x) = \varphi p^{-1}(x)$ resp. $\beta(x) \supset \varphi p^{-1}(x)$. Fixed points of β can then be obtained from coincidence points of (φ, p) since $\mathrm{Fix}(\beta) \supset p\,(\mathrm{Coinc}(\varphi, p))$. The existence theorems in the literature (cf. [4], and its informative bibliography) often assume that p is a Vietoris-map (i.e. proper, with acyclic fibres). Then $p^*: hB \to hE$ is isomorphic in Čech-cohomology h, and the Lefschetz trace of $(p^*)^{-1}\varphi^*: hB$

$\to hB$ can be used to detect fixed points of β. This is clearly related to our theorem 2.1. It appears less general than 2.1 because 2.1 makes no acyclicity-assumption (but if $p: E \to B$ is Vietoris and $D_\varphi = D_g = E$ then $t_g = (p^*)^{-1}$). On the other hand, it has a more general aspect than 2.1 because it doesn't assume an actual fibration (or ENR_B), only a "cohomology fibration" (with "pointlike" fibres). This comparison suggests a common generalization, namely to general *cohomology fibrations* $p: E \to B$ with suitable compactness and ANR-properties. The main step for such a program would be to construct transfer homomorphisms $t_g: hE \to hB$ for proper (co-)homology fibrations. This is an interesting problem in itself but may involve a fair amount of technicalities; for some applications in coincidence theory it could perhaps be bypassed by directly generalizing 3.5 to cohomology fibrations.

(3.8) *Remarks.* If one is primarily interested in coincidence points of $B \xleftarrow{p} E \xrightarrow{\varphi} B$ the methods of this paper can be of help but they are not entirely adequate, not even when generalized as suggested in 3.7. The point is that they are not going after Coinc (φ, p) itself, but rather after the intersection of Coinc (φ, p) with Fix (g). It should be possible to measure Coinc (φ, p) itself, in terms of (co-)homology invariants. If B is manifold then one can use $(\varphi, p)^* (\tau)$, where τ is the Thom-class of the diagonal of $B \times B$. For products $E = Y \times B$, or fibrations as in 3.5, one can define an invariant κ in $\oplus_j (H^j Y \otimes H_j B)$. It seems plausible that this can be adapted to rather general $B \xleftarrow{p} E \xrightarrow{\varphi} B$, at least if B is ENR. But one would expect the invariant to be hard to compute — harder than $J(\varphi, p)$ anyway.

Instead of intersecting Coinc (φ, p) with sets of the form Fix (g) one could probably mimic this process on a (co-)homology level and intersect with other classes than those of the form $\{ \mathrm{Fix}(g) \}$. For instance, in the product case it would presumably amount to taking scalar-products of κ with elements in $H_j Y \otimes H^j B$. Again, one would expect that these numbers are harder to deal with than $J(\varphi, g)$. On the other hand, it seems quite possible that the traces $\Lambda(h)$ in [5], or those of [6] could be obtained in this way.

REFERENCES

[1] Dold, A. *Lectures on Algebraic Topology*. Berlin-Heidelberg-New York, Springer 1972.

[2] —— The Fixed Point Transfer of Fibre-Preserving Maps. *Math. Z. 148* (1976), pp. 215-244.

[3] —— Fixed Point Properties of Product Spaces (in Russian). *Uspehi Mat. Nauk XXXI* (1976), pp. 89-94; *Russian Math. Surv. 31* (1976). pp. 56-61.

[4] Górniewicz, L. Homological Methods in Fixed Point Theory of Multivalued Maps. *Diss. Math. (Rozpr. Mat.) 129* (1976), 71 pp.

[5] Halpern, B. A General Coincidence Theory. *To appear in Pac. J. Math.* (1977).

[6] O'Neill, B. Induced Homomorphisms for Set-valued Maps. *Pac. J. Math. 7* (1957), pp. 1179-1184.

(Reçu le 27 juin 1977)

Albrecht Dold

Mathematisches Institut der Universität
D-6900 Heidelberg

vide-leer-empty