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The set  $T$  is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface  $R$  or of a Fuchsian group  $G$  has a canonical embedding in the space  $T$ . See, for example, [3].

It is natural to ask if there exist relations, other than (1), between  $S$  and  $T$  as subsets of  $B_2$ . Compactness results for conformal mappings show that  $S$  is closed in  $B_2$ . Hence Bers asked in [2] and [3] if one can characterize  $S$  in terms of  $T$  as follows.

QUESTION. *Is  $S$  the closure of  $T$ ?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a  $\varphi$  in  $S$  which does not lie in the closure of  $T$ .*

On the other hand, we have the following characterization of  $T$  in terms of  $S$ . See [4].

THEOREM 2.  *$T$  is the interior of  $S$ .*

## 2. REFORMULATIONS IN THE PLANE

A set  $E \subset \overline{\mathbb{C}}$  is said to be a *quasiconformal circle* if there exists a quasiconformal mapping  $f$  defined in  $\overline{\mathbb{C}}$  which maps the unit circle  $\{z: |z| = 1\}$  onto  $E$ .

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains  $D$ .

THEOREM 3. *There exists a simply connected domain  $D$  and a positive constant  $\delta$  such that  $f(D)$  is not bounded by a quasiconformal circle whenever  $f$  is conformal in  $D$  with  $\|S_f\|_D \leq \delta$ .*

THEOREM 4. *A simply connected domain  $D$  is bounded by a quasiconformal circle if and only if there exists a positive constant  $\delta$  such that  $f$  is univalent in  $D$  whenever  $f$  is meromorphic in  $D$  with  $\|S_f\|_D \leq \delta$ .*

We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a  $\varphi \in S$  and a  $\delta > 0$  such that  $\|\psi - \varphi\| > \delta$  for all  $\psi \in T$ . Choose  $g$  conformal in  $L$  with  $S_g = \varphi$ , let  $D = g(L)$  and suppose that  $f$  is conformal in  $D$  with  $\|S_f\|_D \leq \delta$ . Then  $h = f \circ g$  is conformal in  $L$ ,

$$(2) \quad S_h = (S_f \circ g)(g')^2 + S_g$$

by the composition law for the Schwarzian derivative, and hence  $\psi = S_h \in S$  with

$$\|\psi - \varphi\| = \|S_h - S_g\|_L = \|S_f\|_D \leq \delta.$$

Thus  $\psi \notin T$ ,  $h$  does not have a quasiconformal extension to  $\overline{\mathbf{C}}$ , and  $\partial f(D) = \partial h(L)$  is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let  $\varphi = S_g$  where  $g$  is any conformal mapping of  $L$  onto  $D$ , and choose any  $\psi \in S$  with  $\|\psi - \varphi\| \leq \delta$ . Then  $\psi = S_h$  where  $h$  is conformal in  $L$ ,  $f = h \circ g^{-1}$  is conformal in  $D$  and from (2) we obtain

$$\|S_f\|_D = \|S_h - S_g\|_L = \|\psi - \varphi\| \leq \delta.$$

Hence  $\partial h(L) = \partial f(D)$  is not a quasiconformal circle,  $h$  does not have a quasiconformal extension to  $\overline{\mathbf{C}}$  and  $\psi \notin T$ . Thus the distance from  $\varphi$  to  $T$  is at least  $\delta$  and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

**THEOREM 5.** *There exists a simply connected domain  $D$  and a positive constant  $\delta$  such that  $f(D)$  is not a Jordan domain whenever  $f$  is conformal in  $D$  with  $\|S_f\|_D \leq \delta$ .*

### 3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

**DEFINITION.** *We say that an open arc  $\alpha$  in  $\mathbf{C}$  is a  $b$ -spiral from  $z_1$  onto  $z_2$  if  $\alpha$  has the representation*

$$z = (z_1 - z_2)r(t)e^{it} + z_2, \quad 0 < t < \infty,$$

where  $r(t)$  is positive and continuous with