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RECENT PROGRESS IN THE THEORY OF MINIMAL SURFACES [1]

by E. Bombieri

I. Introduction

In this talk I will report on some recent results in the theory of minimal surfaces. Many of them belong to the theory of higher dimensional minimal varieties and all of them are related to the point of view of Geometric Measure Theory and the Calculus of Variations. The important approach to the various aspects of the 2-dimensional Plateau problem provided by harmonic maps and the Hilbert space setting, will not be treated here. I should also stress the fact that this report is not and does not intend to be a survey of all important achievements of the last years, but rather its purpose is to present a few recent results connected with the central problems of the theory, namely existence, uniqueness and regularity of solutions to the Plateau problem from the point of view of the Calculus of Variations.

II. Currents and Varifolds

Let U be an open set of $\mathbf{R}^n$ and let T be a distribution on smooth differential m-forms φ with compact support in U. The boundary of T is the distribution defined by $(\partial T)(\psi) = T(d\psi)$ where d is the exterior differential; clearly ∂T is a distribution on $(m\text{-}1)$-forms. If T and ∂T are continuous with respect to the L^∞ topology on forms, one says that T is locally normal, and if in addition T has compact support in U one says that T is normal. Normal currents form a Banach space in the following way. Let $M(\varphi)$ be a norm on m-forms, and let $M(T)$ be the dual norm

$$M(T) = \sup \{T(\varphi);\ M(\varphi) \leqq 1\} ;$$

then $N(T) = M(T) + M(\partial T)$ is a norm in the space of normal currents. There is a very special norm on forms, called comass, such that the dual norm, called mass, coincides with m-dimensional area in case T is integra-

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[1]) This article has already been published in *Contributions to Analysis*, papers communicated to an international Symposium in honour of A. Pfluger, ETH Zürich, April 1978. Monographie de l'*Ens. Math.* N° 27, Genève 1979.

tion on an m-dimensional simplex. It follows that if T is integration on an m-dimensional oriented compact manifold V, then its mass is $M(T) = m$-dimensional area of V.

Not all currents are obtained by integration on sets. An integral chain $\gamma = \Sigma n_\sigma \sigma$, where $n_\sigma \in \mathbf{Z}$ and σ are simplexes, determines in a natural fashion a current $\gamma(\varphi) = \Sigma n_\sigma \int_\sigma \varphi$; if f: spt $\gamma \to U$ is a Lipschitz map, then one can define $f_{\#} \gamma$ by means of $f_{\#} \gamma(\varphi) = \gamma(f^{\#} \varphi)$. Now a current T is rectifiable if it can be approximated in the mass norm by currents of type $f_{\#} \gamma, \gamma$ a finite polyhedral chain. If both T and ∂T are rectifiable, one says that T is an integral current. Integral currents are the appropriate generalization of the notion of oriented manifold with boundary, as far as integration of differential forms is concerned. Now the main point is: with respect to a certain very weak notion of convergence (the flat convergence) one has:

(a) a closure theorem, to the effect that if a limit of integral currents is normal, the limit still is an integral current;

(b) a compactness theorem, to the effect that bounded sets of integral (or normal) currents are precompact;

(c) an approximation theorem, to the effect that integral currents can be approximated in the strong norm by smooth C^1 deformations near the identity of suitable integral chains.

Only one warning: the results (a), (b) above hold only if we consider currents with compact support $\subset K$, where K is a compact Lipschitz neighborhood retract (for example K is convex or K has C^2 boundary). Also, the approximation theorem, although it says that integral currents are almost a countable union of C^1 manifolds, does not imply anything about the support of the current. For example, let $z_1, z_2, \ldots, z_m, \ldots$ be a countable dense subset of a compact set $K \subset \mathbf{R}^2$, let C_m be the circle $|z - z_m| = 2^{-m}$ with the usual orientation and let $T = \sum_m \int_{C_m}$. Then T is an integral current, but clearly spt $(T) \supset K$.

Since the mass $M(T)$ is lower semicontinuous on integral currents, one can use the closure and compactness theorems to obtain a solution to Plateau's problem in the following form. Let X be an integral current with support in a fixed nice compact set K; then there is an integral current T of least mass in the set $\{T; \partial T = \partial X, \text{ spt } T \subset K\}$. In fact, one simply takes a minimizing sequence T_i such that $\partial T_i = \partial X$, $M(T_i) \to \inf$ and takes a weak limit $T = \lim T_{i_k}$ on a suitable subsequence. Now $\partial T = \partial X$ and T is integral with spt $T \subset K$, by the compactness theorem, it is also obvious that $M(T) = \liminf M(T_i)$, by lower semicontinuity of mass.

The question arises to what extent this is a satisfactory solution to the problem: among all surfaces with a given boundary, find one with least area. This gives rise to the regularity problem, that is showing that the solutions thus found are indeed manifolds or manifolds outside a small singular set.

The theory of normal and integral currents, as developed by Federer and Fleming [F-F] in their fundamental paper of 1960, is essentially a theory of chains with real or integer coefficients, which has both all reasonable properties of algebraic topology and which at the same time yields reasonable spaces for the purpose of the calculus of variations. However, it is not entirely suitable to study the actual soap films which come out in physical experiments. One difficulty is because of orientation; for example, a Möbius band usually arises as a soap film off a wire which approaches a doubly covered circle. This difficulty can be overcome by working with currents with finite abelian coefficient group, for example with mod 2 coefficients. On the other hand, it became increasingly clear that in order to get a theory suitable for describing physical experiments one had to work in a more set theoretic fashion and give up the useful notion of boundary operator. A convenient theory is the theory of varifolds by Almgren. A varifold is simply a Radon measure on the Grassmann bundle of the space. An appropriate notion of rectifiable and integral varifold is developed, and analogs of the closure, compactness and approximation theorems can be obtained. There are some important differences however and it turns out that currents and varifolds complement each other in several respects.

We end this section by referring to Federer, [FH 1] Ch. IV and Almgren, [AF 1] for precise definitions and proofs of the basic properties of currents and varifolds. All these concepts can be extended to ambient spaces different from euclidean space and in particular to Riemannian manifolds.

III. Recent progress on existence problems

One of the great successes of the Calculus of Variations in the Large has been the proof of existence of closed geodesics on smooth compact Riemannian manifolds. The following striking result is a two dimensional extension.