Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 25 (1979)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: OLD AND NEW PROBLEMS AND RESULTS IN COMBINATORIAL NUMBER THEORY: van der WAERDEN'S THEOREM AND RELATED TOPICS

Autor: Erdös, P. / Graham, R. L.

Kapitel: 1. Introduction

DOI: https://doi.org/10.5169/seals-50387

OLD AND NEW PROBLEMS AND RESULTS IN COMBINATORIAL NUMBER THEORY: van der WAERDEN'S THEOREM AND RELATED TOPICS

by P. Erdös and R. L. Graham

1. Introduction

The present paper represents essentially a chapter in a forthcoming "Monographie" in the *l'Enseignement Mathématique* series [1]) with the title "Old and new problems and results in combinatorial number theory" by the above authors. Basically we will discuss various problems in elementary number theory, most of which have a combinatorial flavor. In general we will avoid classical problems, just mentioning references for the interested reader. We will almost never give proofs but on the other hand we will try to give as exact references as we can. We will restrict ourselves mostly to problems on which we worked for two reasons: (i) In order not to make the paper too long; (ii) We may know more about them than the reader.

Both the difficulty and importance of the problems discussed are very variable—some are only exercises while others are very difficult or even hopeless and may have important consequences or their eventual solution may lead to important advances and the discovery of new methods. Some of the problems we think are difficult may turn out to be trivial after all —this has certainly happened before in the history of the world with anyone who tried to predict the future. Here is an amusing case. Hilbert lectured in the early 1920's on problems in mathematics and said something like this—probably all of us will see the proof of the Riemann hypothesis, some of us (but probably not I) will see the proof of Fermat's last theorem, but none of us will see the proof that $2^{\sqrt{2}}$ is transcendental. In the audience was Siegel, whose deep research contributed decisively to the proof by Kusmin a few years later of the transcendence of $2^{\sqrt{2}}$. In fact shortly thereafter Gelfond and a few weeks later Schneider independently proved that α^{β} is transcendental if α and β are algebraic, β is irrational and $\alpha \neq 0, 1$.

---

[1]) *Monographie N° 28 de l'Enseignement Mathématique*, to appear in 1980.

Thus, we hope the reader will forgive us if some (not many, we hope) of the problems turn out to be disappointingly simple.

The chapter titles in the "Monographie" will be:

I. Van der Waerden's theorem and related topics;
II. Covering congruences;
III. Unit fractions;
IV. Bases and related topics;
V. Completeness of sequences and related topics;
VI. Irrationality and transcendence;
VII. Diophantine problems;
VIII. Miscellaneous problems;
IX. Remarks on an earlier paper.

The "earlier paper" referred to in IX is the problem collection of Erdös, "Quelques problèmes de la théorie des nombres", *Monographie de l'Enseignement Math. No. 6* (1963), 81-135. In IX, we give the current status (to the best of our knowledge) on all the problems which appeared there.

2. Van der Waerden's Theorem and related topics

Denote by $W(n)$ the smallest integer such that if the (positive) integers not exceeding $W(n)$ are partitioned arbitrarily into two classes, at least one class always contains an arithmetic progression (A.P.) of length n. The celebrated theorem of van der Waerden [Wa (27)], [Wa (71)], [Gr-Ro (74)] shows that $W(n)$ exists for all n but all known proofs yield upper bounds on $W(n)$ which are extremely weak, e.g., they are not even primitive recursive functions of n. In the other direction, the best lower bound currently available (due to Berlekamp [Ber (68)]) is

$$W(n+1) > n \cdot 2^n$$

for n prime. It would be very desirable to know the truth here. The only values of $W(n)$ known (see [Chv (69)], [St-Sh (78)]) at present are:

$$W(2) = 3, \quad W(3) = 9, \quad W(4) = 35, \quad W(5) = 178.$$

Recent results of Paris and Harrington [Par-Har (77)] show that certain combinatorial problems with a somewhat similar flavor (in particular, being