Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 25 (1979)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ACYCLIC MAPS
Autor: Hausmann, Jean-Claude / Husemoller, Dale
Kapitel: §6. ACYCLIC MAPS INTO A GIVEN SPACE
DOI: https://doi.org/10.5169/seals-50372

We prove $\tilde{X} \to F$ is a homotopy equivalence with the same argument used in (5.6) to show P_k implies H_k. Since F is also the fibre of $X_N^+ \to [B\pi_1(X)]_N^+$ we have proved the theorem.

(5.8) *Remark.* Using (5.1), we see that for an acyclic map $f : X \to Y$ which is k-simple for all $k \geqq 2$, the homotopy groups $\pi_*(Y)$ can be computed in terms of $\pi_*(X)$ and $\pi_*(B\pi_1(X)_N^+) \cong \pi_*(BN)^+$ for $i \geqq 2$. Some computations of $\pi_*(BN^+)$ for a certain perfect group N can be found for instance in [H, Chapter 7].

§ 6. Acyclic maps into a given space

In this section we study acyclic maps $f: X \to Y$ into a fixed space Y. Two such map $f: X \to Y$ and $f' : X' \to Y$ are called equivalent provided there is a homotopy equivalence $h : X \to X'$ with $f \simeq f'h$. Let $AC(Y)$ denote the class of equivalence classes of acyclic $f: X \to Y$ over Y where X and Y are CW-spaces.

(6.1) Definition. *An extension data over a space* Y *is a triple* (Φ, i, Φ) *where*

(a) Φ *is an extension* $1 \to N \to G \to \pi_1(Y) \to 1$ *with* N *perfect,*

(b) $i : BG \to BG_N^+$ *is an acyclic map with* $\ker(\pi_1(i)) = N$ *(whose equivalence class is well defined by* (3.5)), *and*

(c) $\phi : Y \to BG_N^+$ *is a* 2-*connected map.*

Two triples of extension data (Φ, i, ϕ) and (Φ', i', ϕ') are called equivalent provided there exists an isomorphism $g : G \to G'$ making the following diagrams commutative (up to homotopy for the second one).

$$\begin{array}{ccc} G & \xrightarrow{g} & G' \\ & \searrow \quad \swarrow & \\ & \pi_1(Y) & \end{array} \qquad \begin{array}{ccc} BG & \xrightarrow{Bg} & BG' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ BG_N^+ & \xrightarrow{Bg+} & B(G')_N^+ \\ & {\scriptstyle \phi}\nwarrow \quad \nearrow{\scriptstyle \phi'} & \\ & Y & \end{array}$$

where $N' = g(N)$ and Bg^+ is the unique homotopy equivalence determined by g with (3.1).

We denote by $ED(Y)$ the class of equivalence classes of extension data.

(6.2) DEFINITION. *The data map ρ is the function $\rho : AC(Y) \to ED(Y)$ which assigns to an acyclic map $f: X \to Y$ the class $\rho(f) = (\Phi, i, \phi)$ of extension data defined as follows:*

(a) *Φ is the extension* $1 \to \ker \pi_1(f) \to \pi_1(X) \to \pi_1(Y) \to 1$.

(b) (c) *With the well defined $j : X \to BG$ for $G = \pi_1(X)$ we form the cocartesian diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & BG \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{\phi} & Y \underset{X}{\cup} BG \end{array}$$

Since f is acyclic, i is acyclic, and since $\pi_1(j)$ is an isomorphism, $\ker(\pi_1(i)) = N$. Thus $Y \cup_X BG$ is BG_N^+ up to equivalence.

Now we have to check that the map $\phi : Y \to Y \cup_X BG = BG_N^+$ is 2-connected. Since $\pi_1(j)$ is an isomorphism, $\pi_1(\phi)$ is also an isomorphism. The fact that $\pi_2(\phi)$ is surjective comes from the diagram.

$$\begin{array}{ccccccc} \pi_2(Y) & \xleftarrow{\sim} & \pi_2(\tilde{Y}) & \xrightarrow{\sim} & H_2(\tilde{Y}) & \xleftarrow{\sim} & H_2(\tilde{X}_N) \\ {\scriptstyle \pi_2(\phi)}\downarrow & & \downarrow{\scriptstyle \pi_2(\tilde{\phi})} & & \downarrow & \swarrow & \\ \pi_2(BG_N^+) & \xleftarrow{\sim} & \pi_2(BN^+) & \xrightarrow{\sim} & H_2(N) & & \end{array}$$

The surjectivity on the right is a classical result of Hopf which follows easily from the Serre spectral sequence of the fibration $\tilde{X} \to \tilde{X}_N \to BN$.

Now using (2.5) a simple argument, left to the reader, shows that $\rho : AC(Y) \to ED(Y)$ is well defined.

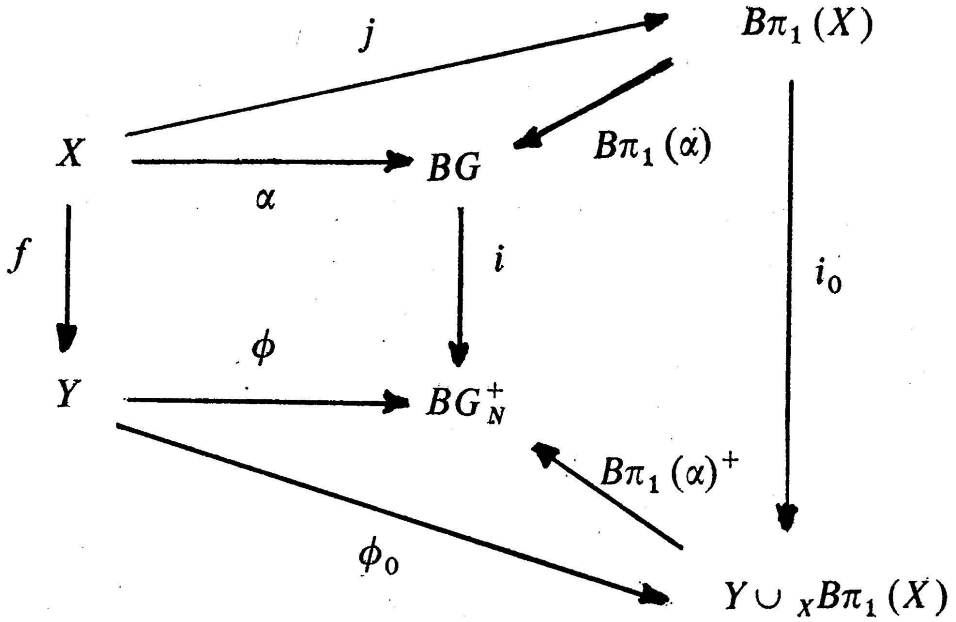
(6.3) THEOREM. *Let Y be a CW-space. The map $\rho : AC(Y) \to ED(Y)$ surjective and its restriction to the subclass $AC_S(Y)$ of $AC(Y)$ of $f: X \to Y$ which are k-simple for all $k \geqq 2$ is a bijection.*

Proof. To show ρ is surjective, consider extension data (Φ, i, ϕ) and form the cartesian square

$$\begin{array}{ccc} X = Y \times_T BG & \xrightarrow{\alpha} & BG \\ f \downarrow & & \downarrow i \\ Y & \xrightarrow{\phi} & BG_N^+ = T \end{array}$$

Now f is acyclic by (2.2), and since its fiber is the same as i, we deduce by (5.2) that f is k-simple for all $k \geqq 2$.

Next, let $\rho(f) = (\Phi_0, i_0, \phi_0)$ and we show this extension data is equivalent to (Φ, i, ϕ). Using the homotopy exact sequences for $X \to Y$ and $BG \to BG_N^+$ and the fact that ϕ is 2-connected, we deduce from the five lemma that $\pi_1(\alpha) : \pi_1(X) \to G$ is an isomorphism. The following diagram shows that (Φ_0, i_0, ϕ_0) is equivalent to (Φ, i, ϕ) and ρ is surjective.



Now, if $f : X \to Y$ is an acyclic map which is k-simple for all $k \geqq 2$ and with $\rho(f) = (\Phi, i, \phi)$, then we form the following commutative diagram.

$$\begin{array}{ccccc} X & \xrightarrow{d} & Y \times_T BG & \longrightarrow & BG \\ & {}_f\searrow & \downarrow f_0 & & \downarrow i \quad (G = \pi_1(X)) \\ & & Y & \xrightarrow{\phi} & BG_N^+ \end{array}$$

As we have seen in the proof the surjectivity of ρ, the map f_0 is acyclic and k-simple for $k \geqq 2$. The map d induces an isomorphism on the fundamental groups and on homology with $\mathbf{Z}\,\pi_1(Y)$ twisted coefficients. By (5.3), the map d is a homotopy equivalence. This proves that the acyclic map f is equivalent to the induced map f_0. Thus ρ restricted to $AC_S(U) \to ED(Y)$ is a bijection.

(6.4) *Remark.* This theorem leaves open the question of the fibres of the function.

$$\rho:\ AC(Y) \to ED(Y).$$

In the next theorem we factor an acyclic map by ones having simplicity properties.

(6.5) *Remark.* In theorem (6.3), if one fixes an extension $\Phi : 1 \to N \to G \to \pi_1(Y) \to 1$, then the same proof permits us to classify acyclic maps $f : X \to Y$ which are k-simple for $k > 2$ together with an identification $d : \pi_1(X) \to G$ such that $\Phi d = \pi_1(f)$. The objects of $ED(Y)$ have to be replaced by couples (i, ϕ) where $i : BG \to BG_N^+$ is as above and $\phi : Y \to BG_N^+$ is 2-connected with the following diagram commuting up to homotopy.

$$\begin{array}{ccccc} & & B\pi_1(Y) & \xrightarrow{B\Phi} & BG \\ & \nearrow & & \searrow{\scriptstyle B\Phi^+} \qquad \swarrow{\scriptstyle i} & \\ Y & & \xrightarrow{\quad\phi\quad} & BG_N^+ & \end{array}$$

This is what is done implicitely in [H, Sections 2 and 4]. Observe that we are dealing here with classes which are sets.

(6.6) LEMMA. *Let X be a CW-space and N a perfect normal subgroup of $\pi_1(X)$. Let $X \to P_nX$ denote the nth stage of the Postnikov decomposition of X. Then for all $n \geqq 1$ we have that*

(1) $\pi_j(X_N^+) \to \pi_j((P_nX)_N^+)$ *is an isomorphism for $j \leqq n$ and an epimorphism for $j = n+1$, and*

(2) $\pi_j(A\tilde{X}_N) \to \pi_j(A(P_n\tilde{X}_N))$ *is an isomorphism for $j \leqq n$ and an epimorphism for $j = n+1$.*

Proof. Consider the following homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} T & \longrightarrow & A\tilde{X}_N & \longrightarrow & A(P_n\tilde{X}) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & \tilde{X}_N & \longrightarrow & P_n\tilde{X}_N \\ \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & (\tilde{X}_N)^+ & \longrightarrow & (P_n\tilde{X}_N)^+ . \end{array}$$

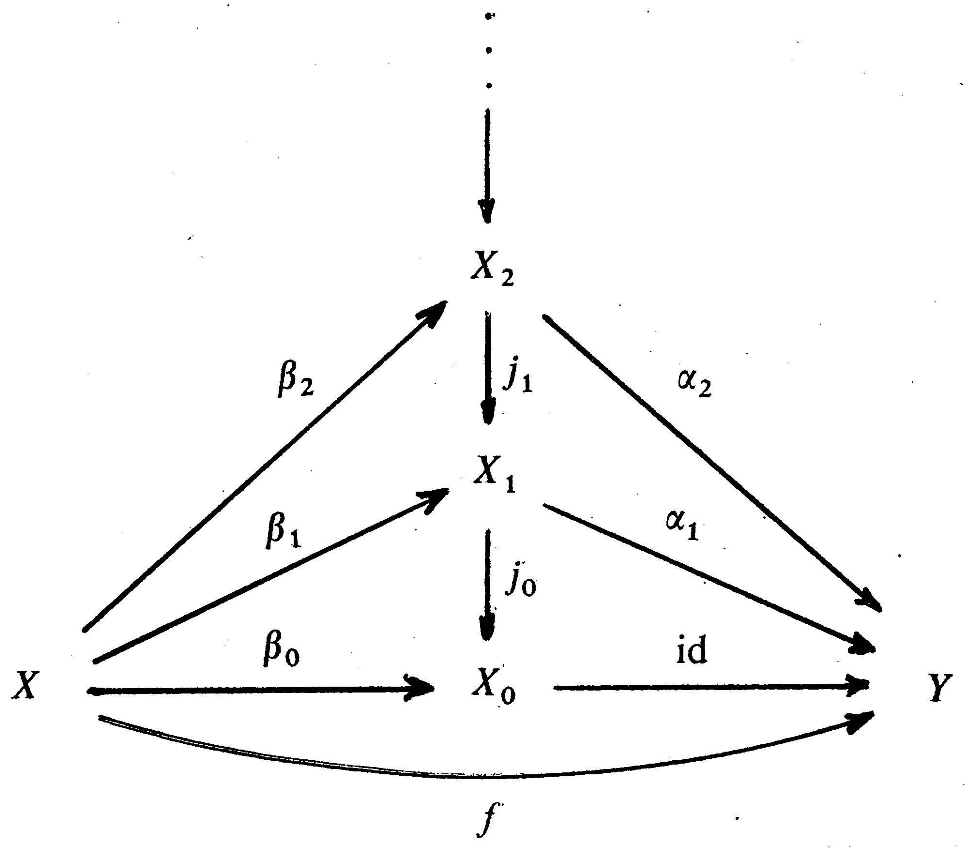
Clearly $\pi_i(F) = 0$ for $i \leqq n+1$. The spaces $\tilde{X}_N$ and $P_n \tilde{X}_N$ have the same $(n+1)$-skeleton and the same can be assumed for $\tilde{X}_N^+$ and $(P_n \tilde{X}_N)^+$. Hence $\pi_i(G) = 0$ for $i \leqq n+1$. Now (1) follows because G is the fibre of $X_N^+ \to (P_n X)^+$.

By comparing Serre spectral sequences, we obtain the surjectivity of

$$H_0(N, H_{n+1}(F)) \to H_0(N, H_{n+1}(G)) = H_{n+1}(G) = \pi_{n+1}(G).$$

Thus $\pi_j(T) = 0$ for $j \leqq n$ and (2) follows.

(6.7) THEOREM. *Let $f: X \to Y$ be a map between CW-spaces. Then there is a factorization*



such that β_i is i-connected and α_i is an acyclic map which is k-simple for $k > i$.

Such a decomposition is unique up to a homotopy equivalence.

Proof. The ith stage X_i is defined by the cartesian diagram

$$\begin{array}{ccc} Y \times_T P_i(X) & \longrightarrow & P_i X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & (P_i X)_N^+ = T \end{array}$$

where $N = \ker(\pi_1(X) \to \pi_1(Y))$. By (6.6) the map β_i is i-connected since the fiber of the two vertical arrows is $A(P_n\tilde{X})_N$. Now by (5.4) we see that α_i is simple for $k > i$.

For two decompositions (X'_i) and (X''_i) of $f : X \to Y$ satisfying the above conditions, we have $P_i X'_i = P_i X''_i$ and both X'_i and X''_i map into X_i, constructed above, such that the resulting diagrams are homotopy commutative. The connectivity of the β_i and (5.1) shows that these maps are all homotopy equivalences. This proves the theorem.

(6.8) *Remarks.* This theorem (6.7) coincides with the Dror results for Y a point [D1, Theorem 1.3] and $Y = S^n$ [D2]. An interesting problem is to describe the ith stage X_i in terms of invariants of X_{i-1} as in [D1] and [D2]. (See the footnote in the introduction.)

Appendix — Simplicity properties of fibers

In the proof of (5.4) we used the fact that for a fibration $F \to E \xrightarrow{f} B$ the action of $\pi_1(F)$ on $\mathrm{Im}(\partial : \pi_{k+1}(B) \to \pi_k(F))$ is trivial. This assertion does not seem to be in the literature so we include a proof here.

We extend the mapping sequence of the fibration f to $\Omega B \to F \to E \xrightarrow{f} B$ and study F as the total space of a principal fibration with fibre the H-space ΩB. If G is an H-space, then $\pi_1(G)$ acts trivally on $\pi_*(G)$ because the covering transformations $\tilde{G} \to G$ on the universal covering $\tilde{G}$ of G are homotopic to the identity. This is proved by lifting a loop to a path in $\tilde{G}$ and using the H-space structure on $\tilde{G}$ to deform the identity along this path to the covering transformation defined by the homotopy class of the loop. Recall that a principal fibration is induced from $G \to E_G \to B_G$ up to fibre homotopy equivalence.

(A.1) Proposition. *Let* $G \to X \xrightarrow{\pi} Y$ *be a principal fibration with fibre* G *acting on* X. *Then we have:*

(a) $\mathrm{im}(\pi_1(G) \to \pi_1(X))$ *acts trivially on* $\pi_*(X)$, *and*

(b) $\pi_1(X)$ *acts trivially on* $\mathrm{im}(\pi_*(G) \to \pi_*(X))$.

Proof. For (a) we have the following commutative diagram induced by a covering transformation $T : \tilde{G} \to \tilde{G}$.