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ON LIE'S HIGHER SPHERE GEOMETRY

by Jay P. Fillmore

1. Introduction

In this paper we draw together two theories having their roots in the ideas of S. Lie over a century ago: *Lie's higher sphere geometry*, with its famous *line-sphere transformation* [5], and the theory of Lie groups, especially the description of a geometry by global Lie groups [1]. Indeed, not until the 1960s, with the appearance of W. M. Boothby's description of homogeneous contact manifolds [1, 2] and with the appearance of parabolic subgroups, could this connection be established. One can now say, in terms of Lie groups, that the three-dimensional complex line and sphere geometries are isomorphic and that *the real line and sphere geometries are two distinct real forms of one geometry.* Furthermore, the line-sphere transformation gives explicitly the isomorphism of the complex forms.

In Section 2 we summarize the formulation of Boothby's theory for algebraic homogeneous contact manifolds and make some observations about their real forms. The classical contact manifolds of complex co-directions in projective space and of Lie's higher sphere geometry are described in general in terms of this theory in Sections 3 and 4. Finally, in Section 5, the connection with Plücker's line geometry in three dimensions is established, and the line-sphere transformation is brought into perspective. This introduction continues with an overview of F. Klein's formulation of Lie's theory [5, 6], Boothby's theory, and their connection.

To a line in complex projective space P^3 may be assigned Plücker coordinates

$$\xi_1 = p_{12}\,,\ \xi_2 = p_{31}\,,\ \xi_3 = p_{23}\,,$$
$$\xi_4 = p_{03}\,,\ \xi_5 = p_{02}\,,\ \xi_6 = p_{01}\,,$$

[6, §20]. These coordinates satisfy

$$\xi_1\xi_4 + \xi_2\xi_5 + \xi_3\xi_6 = 0\,,$$

[1] This description of Lie's higher sphere geometry in terms of Lie groups answers a question posed in 1965 by S. Sasaki [9, p. 173].

and hence lines in P^3 correspond to points of a quadric Ω^4 in P^5. Two lines in P^3 intersect when their corresponding points on Ω^4 are conjugate. A surface element in P^3, a point and incident plane, becomes the pencil of lines passing through the point and lying in the plane; this corresponds to a line lying in Ω^4. The space of surface elements in P^3 thus corresponds to the space of lines in Ω^4. The projectivities of P^5 which preserve the quadric Ω^4 permute the lines of Ω^4 and hence the surface elements of P^3. Moreover, these projectivities preserve the condition, between two surface elements at infinitesimally adjacent points, that a point of one lies on the plane of the other; hence they are contact transformations of P^3.

To a sphere

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + C = 0$$

in complex Euclidean space E^3, with center at $x = a, y = b, z = c$ and radius

$$r^2 = a^2 + b^2 + c^2 - C,$$

the sign of r corresponding to an "orientation", may be assigned homogeneous coordinates

$$a = \frac{\alpha}{\nu}, \quad b = \frac{\beta}{\nu}, \quad c = \frac{\gamma}{\nu}, \quad r = \frac{\lambda}{\nu}, \quad C = \frac{\mu}{\nu},$$

[6, §25]. These coordinates satisfy

$$\alpha^2 + \beta^2 + \gamma^2 - \lambda^2 - \mu\nu = 0,$$

and hence oriented spheres in E^3 correspond to certain points of a quadric Ψ^4 in P^5; if spheres which are points or planes or which have centers at infinity are included, all points of Ψ^4 are obtained. Two spheres in E^3 are tangent at a point, orientations taken into account, when their corresponding points on Ψ^4 are conjugate. An "oriented" surface element in E^3, a point and incident oriented plane, becomes the pencil of spheres tangent to the plane at the point; this corresponds to a line lying in Ψ^4. The space of oriented surface elements of E^3 thus corresponds to the space of lines in Ψ^4. The projectivities of P^5 which preserve the quadric Ψ^4 permute the lines of Ψ^4 and hence the oriented surface elements of E^3. Moreover, these projectivities are contact transformations of E^3.

The line-sphere transformation, discovered by Lie, is given by

$$\begin{aligned} \xi_1 &= \alpha + \sqrt{-1}\,\beta, & \xi_4 &= \alpha - \sqrt{-1}\,\beta, \\ \xi_2 &= \gamma + \lambda, & \xi_5 &= \gamma - \lambda, \\ \xi_3 &= \mu, & \xi_6 &= -\nu, \end{aligned}$$

as formulated by Klein [6, §70]. This makes correspond points of the quadric Ω^4 of signature $(+++---)$ and points of the quadric Ψ^4 of signature $(++++--)$. Conjugate points correspond to conjugate points, and a line in one quadric corresponds to a line in the other. Thus, surface elements in P^3 correspond to oriented surface elements in E^3 and this correspondence is a "contact transformation".

Now, classically a *contact transformation* in P^3 or E^3 is a transformation on the 5-dimensional space of surface elements which preserves, up to a non-vanishing multiple, a maximal rank Pfaffian form

$$\omega = dz - pdx - qdy,$$

[6, §63], where the coordinates x, y, z, p, q describe the surface element consisting of the plane

$$z' - z = p(x'-x) + q(y'-y)$$

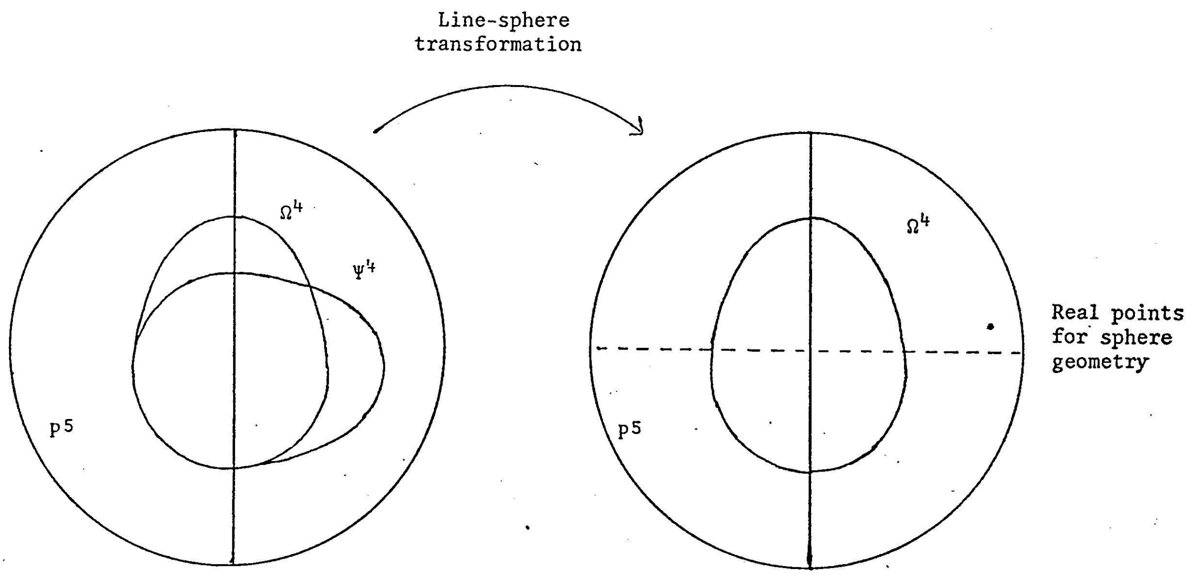
at the point (x, y, z). The condition $\omega = 0$, that at two infinitesimally adjacent points the point of one surface element lies on the plane of the other, is preserved by a contact transformation. The appropriate spaces for the line-sphere transformation are the 5-dimensional spaces of lines in Ω^4 and lines in Ψ^4. Exhibiting the Pfaffian forms and examining the effect of the line-sphere transformation on them may be done systematically by observing that these spaces are *homogeneous.*

Boothby's description of compact homogeneous complex contact manifolds [1, 2; and 7, §2] constructs for each type of simple complex Lie algebra $\mathfrak{g}$: a connected centerless simple Lie groups G having Lie algebra $\mathfrak{g}$, a parabolic subgroup P of G, and a Pfaffian form ω on a principal $\mathbf{C}^*$-bundle over G/P, so that G/P, with ω pulled down by local sections, is a compact complex contact manifold, homogeneous under the identity component G of the group of all its contact automorphisms. Every such contact manifold is so obtained uniquely up to isomorphism. This construction yields, for the classical simple Lie algebras:

A_n	projective cotangent bundle of P^n—the classical space of incident point-hyperplane pairs in P^n,
B_l and D_l	space of lines in a quadric,
C_l	odd-dimensional projective space P^{2l+1},

[1, (7.1)]. The isomorphism $A_3 \simeq D_3$ arises from the description of surface elements in P^3 as lines in Ω^4 by Plücker coordinates. Since the complex quadrics Ω^4 and Ψ^4 both have groups of projectivities of the type D_3, the contact manifolds of line geometry and sphere geometry, when viewed as the spaces of lines in Ω^4 and Ψ^4 respectively, are necessarily *the same*, that is, isomorphic.

When Boothby's description of homogeneous contact manifolds is refined, using J. A. Wolf's theory of complex flag manifolds [8, Ch. I], to include their real forms, line geometry and sphere geometry are *no longer the same*, but, as was classically recognized [6, §25], are obtained from the real forms $PSO\,(3,3;\mathbf{R})$ and $PSO\,(4,2;\mathbf{R})$ of $PSO\,(3,3;\mathbf{C})$ and $PSO\,(4,2;\mathbf{C})$, where the quadratic forms defining these projective special orthogonal groups are those of the quadrics Ω^4 and Ψ^4. Now, $PSO\,(3,3;\mathbf{C})$ and $PSO\,(4,2;\mathbf{C})$ are isomorphic, so the corresponding complex contact manifolds are isomorphic; in fact, these groups, are conjugate in $PSL\,(6;\mathbf{C})$ by the matrix of Klein's description of the line-sphere transformation. Viewed another way, $PSO\,(3,3;\mathbf{R})$ and $PSO\,(4,2;\mathbf{R})$ correspond to two real forms of $PSO\,(3,3;\mathbf{C})$ defined by two complex conjugations. Consequently, *real line geometry and real sphere geometry are two distinct real forms of complex line geometry.* The line-sphere transformation then corresponds to an automorphism of $PSO\,(3,3;\mathbf{C})$ connecting the two complex conjugations.



Real points of P^5. Real points for line geometry