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### 5. THE LINE-SPHERE TRANSFORMATION

The homogeneous contact manifold of co-directions in complex projective space  $P^3$ , obtained from the simple complex Lie algebra of type  $A_3$ , must coincide with that of oriented co-directions in complex Euclidean space  $E^3$ , obtained from the algebra of type  $D_3$ , in view of the isomorphisms  $A_3 \simeq D_3$ . To exhibit this explicitly, we introduce a third homogeneous contact manifold in terms of which both of these can be conveniently described, namely, the space of lines in the quadric  $\Omega^4$  in  $P^5$  of Section 1.

5.1. We carry out the construction of 2.10 for the simple complex Lie algebras of type  $B_l$  and  $D_l$ , making the restriction to type  $D_3$  later.

Let  $\mathfrak{g} = \mathfrak{o}(A; \mathbb{C})$ , complex square matrices  $X$  for which  ${}^tXA + AX = 0$ , where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \quad \text{in case } B_l$$

or

$$A = \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix} \quad \text{in case } D_l,$$

that is, the quadratic form defining  $\mathfrak{g}$  is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

or

$$2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

respectively [4, (16.3) and (16.4)].

We exhibit the details of the construction for the case of  $D_l$ . For  $B_l$  one need only carry along an additional initial row and column in the matrices, as well as the corresponding roots; the conclusions are the same.

Thus  $\mathfrak{g}$  consists of  $2l$  by  $2l$  matrices of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{bmatrix},$$

where  $X_1$  is  $l$  by  $l$  and arbitrary and  $X_2$  and  $X_3$  are  $l$  by  $l$  and skew-symmetric. For Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  take diagonal matrices  $H$  of the form

$$H = \text{diag} (h_1, \dots, h_l \mid -h_1, \dots, -h_l).$$

Let  $\delta_i, i = 1, 2, \dots, l$  be the linear function on  $\mathfrak{h}$  which assigns  $h_i$  to  $H: \delta_i(H) = h_i$ . The roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  are

$$\begin{aligned} \pm \delta_i \pm \delta_j \quad i, j = 1, 2, \dots, l \\ \text{and } i \neq j \end{aligned}$$

and the root vector  $E_\alpha$  corresponding to the root  $\alpha$  is

$$\begin{aligned} E_{\delta_i - \delta_j} &= \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, \quad i \neq j, \\ E_{\delta_i + \delta_j} &= \begin{bmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{bmatrix}, \quad i < j, \\ E_{-\delta_i - \delta_j} &= \begin{bmatrix} 0 & 0 \\ E_{ji} - E_{ij} & 0 \end{bmatrix}, \quad i < j, \end{aligned}$$

where  $E_{ij}$  is the  $l$  by  $l$  matrix with 1 in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column and 0s elsewhere [4, (16.3)]. A system of simple roots is

$$\delta_1 - \delta_2, \delta_2 - \delta_3, \dots, \delta_{l-1} - \delta_l, \quad \text{and} \quad -\delta_1 - \delta_2,$$

(this is not the same choice as in [4, (16.3)]), for which the maximal root is

$$\rho = -\delta_{l-1} - \delta_l,$$

[4, App., Table E]. The Killing form of  $\mathfrak{g}$  is  $\langle X, Y \rangle = (2l-2) \text{tr}(XY)$ , but we replace this with  $\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY)$  for convenience. Then the  $H_\alpha$  in  $\mathfrak{h}$  are given by

$$H_{\pm\delta_i \pm \delta_j} = \text{diag}(0, \dots, 0, \pm 1, 0, \dots, 0, \pm 1, 0, \dots, 0 \mid \text{---}),$$

where the  $\pm 1$ s occur in the  $i^{\text{th}}$  and  $j^{\text{th}}$  entries and the second  $l$  entries are the negatives of the first  $l$  entries. Especially,

$$H_\rho = \text{diag}(0, \dots, 0, -1, -1 \mid 0, \dots, 0, 1, 1).$$

It is now straightforward to determine for which roots  $\alpha$  we have  $\langle H_\rho, H_\alpha \rangle \geq 0$  and find that  $\mathfrak{p}$  in (i) of 2.9 consists of matrices of the form

$$\left[ \begin{array}{c|ccc} & \text{arbitrary} & 0 & 0 \\ & (l-2) \text{ by } (l-2) & \vdots & \vdots \\ & \text{skew-} & & \\ & \text{symmetric} & 0 & 0 \\ & 0 \text{ -----} & 0 & 0 & 0 \\ & 0 \text{ -----} & 0 & 0 & 0 \\ \hline & * \text{ -----} & * & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots \\ & \text{arbitrary} & & 0 & 0 \\ & l \text{ by } l & & * & * \\ & \text{skew-} & & & \\ & \text{symmetric} & & * & * \\ & * \text{ -----} & * & * & * \end{array} \right],$$

where the starred entries are arbitrary.

5.2. The connected centerless simple group  $G = PSO(A; \mathbb{C})$  is transitive on the lines of the quadric  $\Omega^{2l-2}$

$$\xi_1 \xi_{l+1} + \dots + \xi_l \xi_{2l} = 0$$

in  $P^{2l-1}$  by Witt's theorem. The Lie algebra of the isotropy subgroup of the line  $l_0$  joining

$${}^t(0, \dots, 0, 1, 0) \quad \text{and} \quad {}^t(0, \dots, 0, 0, 1)$$

is  $\mathfrak{p}$ . Hence

$$G/P = \text{space of lines in } \Omega^{2l-2}.$$

The element  $W = E_\rho$  of  $\mathfrak{p}$  giving the contact structure on  $G/P$ , as in 2.7, is

$$W = \left[ \begin{array}{c|ccc} & 0 & & 0 \\ & & & \\ \hline & 0 & & \\ & & & 0 \\ & 0 & -1 & \\ & 1 & 0 & \end{array} \right]$$

In general, the construction of 2.10 gives the  $(2n-1)$ -dimensional homogeneous contact manifold of lines in the quadric  $\Omega^{n+1}$  in  $P^{n+2}$ , where  $\Omega^{n+1}$  is

$$\xi_0^2 + 2\xi_1 \xi_{l+1} + \dots + 2\xi_l \xi_{2l} = 0$$

in case  $B_l$  when  $n$  is even,  $n+3 = 2l+1$ , and  $\Omega^{n+1}$  is  $\Omega^{2l-2}$  above in case  $D_l$  when  $n$  is odd,  $n+3 = 2l$ ;  $n \geq 2$ .

The real contact structure on the  $(2n-1)$  dimensional space of lines of  $\Omega^{n+1}$  in real projective space  $P^{n+2}$  is described by viewing all quantities in the foregoing discussion as being real. Especially,  $G_0$  of 2.11 is the one- or two- component centerless group  $PSO(A; \mathbf{R})$  consisting of real contact automorphisms.

5.3. The line joining  $x = {}^t(x_0, x_1, x_2, x_3)$  and  $y = {}^t(y_0, y_1, y_2, y_3)$  in complex projective space  $P^3$  has Plücker coordinates  $p_{ij} = x_i y_j - x_j y_i$ . These coordinates are the coefficients of the bivector  $x \wedge y$  with respect to the basis

$$e_1 \wedge e_2, e_3 \wedge e_1, e_2 \wedge e_3, e_0 \wedge e_3, e_0 \wedge e_2, e_0 \wedge e_1,$$

where  $e_0 = {}^t(1, 0, 0, 0), \dots, e_3 = {}^t(0, 0, 0, 1)$ , and satisfy

$$p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0,$$

[6, §69]. If we set

$$\begin{aligned} \xi_1 &= p_{12}, & \xi_2 &= p_{31}, & \xi_3 &= p_{23}, \\ \xi_4 &= p_{03}, & \xi_5 &= p_{02}, & \xi_6 &= p_{01}, \end{aligned}$$

we have that the lines of  $P^3$  correspond to the points of the quadric  $\Omega^4$

$$\xi_1 \xi_4 + \xi_2 \xi_5 + \xi_3 \xi_6 = 0$$

in  $P^5$ . Two lines of  $P^3$  intersect exactly when their corresponding points on  $\Omega^4$  are conjugate, that is, the line joining these points lies entirely in  $\Omega^4$ .

To a point  $x$  in  $P^3$  we associate all lines of  $P^3$  incident with  $x$  and hence a plane lying in  $\Omega^4$ . To a plane  $u$  in  $P^3$  we associated all lines of  $P^3$  lying in  $u$  and hence a plane lying in  $\Omega^4$ . These two families of planes doubly rule  $\Omega^4$ . To a surface element or co-direction in  $P^3$ , that is, a point  $x$  and incident plane  $u$ , is then associated all lines of  $P^3$  lying in  $u$  and incident with  $x$ . In  $\Omega^4$  this corresponds to the intersection of the planes corresponding to  $x$  and  $u$  and is a line. Hence, the 5-dimensional spaces of co-directions in  $P^3$  and lines in  $\Omega^4$  correspond naturally.

Note that the co-direction in  $P^3$  consisting of the point  $x_0 = {}^t(1, 0, 0, 0)$  and the incident plane  $u_0: x_3 = 0$  in 3.2 corresponds to the line  $l_0$  of  $\Omega^4$  joining the points  ${}^t(0, 0, 0, 0, 1, 0)$  and  ${}^t(0, 0, 0, 0, 0, 1)$  in 5.2. For, to the co-direction  $(x_0, u_0)$  is associated all lines of  $P^3$  joining  $x_0$  and a point  $y = {}^t(y_0, y_1, y_2, 0)$  of  $u_0$ ; such a line has Plücker coordinates

$$\begin{aligned} \xi_1 &= 0, & \xi_2 &= 0, & \xi_3 &= 0, \\ \xi_4 &= 0, & \xi_5 &= y_2, & \xi_6 &= y_1, \end{aligned}$$

and corresponds to a point of  $\Omega^4$  lying on  $l_0$ .

The projectivity  $g$  in  $PSL(4; \mathbf{C})$  permutes the lines of  $P^3$  by  $x \wedge y \rightarrow gx \wedge gy$ , a projectivity of  $P^5$  which preserves  $\Omega^4$ . In this way one obtains the isomorphism  $A_3 \simeq D_3$ :

$$PSL(4; \mathbf{C}) \simeq PSO(A; \mathbf{C}), \quad A = \begin{bmatrix} 0 & 1_3 \\ 1_3 & 0 \end{bmatrix},$$

[4, (25.8.4')]. The spaces of co-directions in  $P^3$  and lines in  $\Omega^4$  are homogeneous under  $PSL(4; \mathbf{C})$  and  $PSO(A; \mathbf{C})$  respectively; hence the correspondence between these spaces is as homogeneous spaces. In fact, since  $(x_0, u_0)$  and  $l_0$  correspond, their isotropy subgroups, as described in 3.2 and 5.2, correspond under the isomorphism.

From the isomorphism of the groups, we obtain the isomorphism of the Lie algebras  $\mathfrak{sl}(4; \mathbf{C}) \simeq \mathfrak{o}(A; \mathbf{C})$ , where  $X$  in  $\mathfrak{sl}(4; \mathbf{C})$  is sent to the linear transformation  $x \wedge y \rightarrow (Xx) \wedge y + x \wedge (Xy)$  in  $\mathfrak{o}(A; \mathbf{C})$ . With  $X = (a_{ij})$ ,  $i, j = 0, 1, 2, 3$ , the matrix of this transformation with respect to the basis  $e_i \wedge e_j$  is

$$\begin{bmatrix} a_{11} + a_{22} & -a_{23} & -a_{13} & 0 & a_{10} & -a_{20} \\ -a_{32} & a_{11} + a_{33} & -a_{12} & -a_{10} & 0 & a_{30} \\ -a_{31} & -a_{21} & a_{22} + a_{33} & a_{20} & -a_{30} & 0 \\ \hline 0 & -a_{01} & a_{02} & a_{00} + a_{33} & a_{32} & a_{31} \\ a_{01} & 0 & -a_{03} & a_{23} & a_{00} + a_{22} & a_{21} \\ -a_{02} & a_{03} & 0 & a_{13} & a_{12} & a_{00} + a_{11} \end{bmatrix};$$

this describes the isomorphism explicitly. Under this isomorphism, the Lie algebras of the isotropy subgroups of  $(x_0, u_0)$  and  $l_0$ , as in 3.1 and 5.1, correspond. Moreover, the element

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ of } \mathfrak{sl}(4; \mathbf{C})$$

is sent into the element

$$\begin{bmatrix} & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & -1 & \\ 0 & 1 & 0 & \end{bmatrix} \text{ of } \mathfrak{o}(A; \mathbf{C}).$$

Since these are the root vectors for the maximal roots which determine the contact structures, as in 3.4 and 5.2, we conclude:

The 5-dimensional manifolds of co-directions in  $P^3$  and lines in  $\Omega^4$  are isomorphic as algebraic homogeneous contact manifolds.

This isomorphism holds for the real contact manifolds also; cf. 3.5 and 5.2. The real connected centerless groups  $PSL(4; \mathbf{R})$  and  $PSO(A; \mathbf{R})$  are isomorphic; each consists of the elements fixed under complex conjugation of matrix entries.

5.4. The algebraic homogeneous contact manifolds of lines in the quadrics  $\Psi^{n+1}$  and  $\Omega^{n+1}$ , 4.3 and 5.2, are isomorphic since they are both obtained from the simple complex Lie algebra of type  $B_l$  or  $D_l$  by the construction of 2.10. This isomorphism can be exhibited explicitly by means of a contact transformation which reduces to the line-sphere transformation, as described in Section 1, when  $n = 3$ .

Throughout, unprimed quantities refer to  $\Omega^{n+1}$  and primed quantities to  $\Psi^{n+1}$ . Set  $n + 3 = 2l + 1$  or  $2l$  according as  $n$  is even or odd;  $n \geq 2$ .

Thus,

$$G = PSO(A; C), A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix}$$

and

$$G' = PSO(A'; C), A' = \begin{bmatrix} 2 \cdot 1_n & & & 0 \\ \hline & & & \\ & & -2 & 0 & 0 \\ & 0 & 0 & 0 & -1 \\ & & & 0 & -1 & 0 \end{bmatrix}.$$

These are groups of projectivities preserving  $\Omega^{n+1}$  and  $\Psi^{n+1}$ , respectively, in  $P^{n+2}$ .

In case  $n$  is odd, the transformation which we consider is

$$\begin{aligned} \xi_1 &= \alpha_1 + \sqrt{-1} \alpha_2 & \xi_{l+1} &= \alpha_1 - \sqrt{-1} \alpha_2 \\ \xi_2 &= \alpha_3 + \sqrt{-1} \alpha_4 & \xi_{l+2} &= \alpha_3 - \sqrt{-1} \alpha_4 \\ & \vdots & & \vdots \\ \xi_{l-2} &= \alpha_{n-2} + \sqrt{-1} \alpha_{n-1} & \xi_{2l-2} &= \alpha_{n-2} - \sqrt{-1} \alpha_{n-1} \\ \xi_{l-1} &= \alpha_n + \lambda & \xi_{2l-1} &= \alpha_n - \lambda \\ \xi_l &= \mu & \xi_{2l} &= -v. \end{aligned}$$

This is a projectivity of  $P^{n+2}$  which sends the quadric  $\Psi^{n+1}$

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu v = 0$$

into the quadric  $\Omega^{n+1}$

$$2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l} = 0.$$

In case  $n$  is even, the first equation of the transformation is  $\xi_0 = \sqrt{2} \alpha_1$  and the remaining ones are like the above.

As before, we exhibit the details of the calculations for the case of  $n$  odd. For  $n$  even one need only carry along an additional initial row and column in the matrices; the conclusions are unchanged.

The matrix  $T$  of the transformation is



$$T = \begin{bmatrix} B & & & & 0 \\ & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ \bar{B} & & & & 0 \\ & 1 & -1 & 0 & 0 \\ 0 & & & & \\ & 0 & 0 & 0 & -1 \end{bmatrix},$$

where  $B$  is the  $(l-2)$  by  $(2l-4)$  matrix

$$B = \begin{bmatrix} 1 & \sqrt{-1} & & & \\ & & 1 & \sqrt{-1} & 0 \\ & & & & \vdots \\ & 0 & & & \vdots \\ & & & & 1 & \sqrt{-1} \end{bmatrix}$$

and  $\bar{B}$  is its complex conjugate;  $T$  has inverse

$$T^{-1} = \frac{1}{2} \begin{bmatrix} {}^t\bar{B} & 0 & {}^tB & 0 \\ & 1 & 0 & 1 & 0 \\ & 0 & 1 & 0 & -1 & 0 \\ & & 0 & 2 & 0 & 0 \\ & & 0 & 0 & 0 & -2 \end{bmatrix}.$$

By direct calculation we ascertain the following:

- (1)  $A' = {}^tTAT$  and hence  $G' = T^{-1}GT$ .  $G$  and  $G'$  are conjugate, but do not coincide, in  $PSL(n+3; \mathbb{C})$ . As a consequence,  $g' = T^{-1}gT$ .
- (2)  $l'_0 = T^{-1}l_0$ ; the line  $l_0$  in  $\Omega^{n+1}$  joining

$${}^t(0, \dots, 0, 1, 0) \quad \text{and} \quad {}^t(0, \dots, 0, 0, 1)$$

is sent to the line  $l'_0$  of  $\Psi^{n+1}$  joining

$${}^t(0, \dots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \dots, 0, 1 \mid -1, 0, 0).$$

Hence their isotropy subgroups, as in 5.2 and 4.3 are conjugate:  $P' = T^{-1} P T$ . As a consequence,  $\mathfrak{p}' = T^{-1} \mathfrak{p} T$ .

(3) The Cartan subalgebras of  $\mathfrak{g}$  and  $\mathfrak{g}'$  in 5.1 and 4.4 are conjugate:  $\mathfrak{h}' = T^{-1} \mathfrak{h} T$ . In fact, for

$$H = \text{diag}(h_1, \dots, h_l \mid -h_1, \dots, -h_l)$$

in  $\mathfrak{h}$ , we have

$$T^{-1} H T = \text{diag} \left[ \begin{array}{cc} 0 & \sqrt{-1}h_1 \\ -\sqrt{-1}h_1 & 0 \end{array} \right], \dots, \left[ \begin{array}{cc} 0 & \sqrt{-1}h_{l-2} \\ -\sqrt{-1}h_{l-2} & 0 \end{array} \right],$$

$$\left[ \begin{array}{cccc} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{array} \right]$$

in  $\mathfrak{h}'$ .

4) The elements  $W$  and  $W'$  of the Lie algebras which give the contact structures on  $G/P$  and  $G'/P'$ , as in 5.2 and 4.4, are conjugate:  $W' = T^{-1} W T$ . We conclude:

The  $(2n-1)$ -dimensional manifolds of lines in  $\Omega^{n+1}$  and lines in  $\Psi^{n+1}$  are isomorphic as algebraic homogeneous contact manifolds. The isomorphism is a consequence of the projectivity  $T$  carrying  $\Psi^{n+1}$  into  $\Omega^{n+1}$ .  $T$  sends lines of  $\Psi^{n+1}$  into lines of  $\Omega^{n+1}$  and is a contact transformation.

5.5.  $G_0 = PSO(A; \mathbf{R})$  is a real form of  $G$ ; it consists of the elements of  $G$  fixed under the conjugation  $g \rightarrow \bar{g}$  of  $G$ , complex conjugation of the matrix entries of  $g$ . The Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , as in 5.1, is stable and the maximal root  $\rho = -\delta_{l-1} - \delta_l$  is real. With  $P_0 = G_0 \cap P$ , we obtain from 2.11 the real contact manifold

$$G_0/P_0 = \text{space of lines in } \Omega^{n+1} \text{ in real } P^{n+2},$$

a real form of  $G/P$ ; cf. 5.2. The same remarks apply to the real form  $G'_0 = PSO(A'; \mathbf{R})$  of  $G'$  for the conjugation  $g' \rightarrow \bar{g}'$ . With  $P'_0 = G'_0 \cap P'$ , we obtain the real contact manifold

$$\begin{aligned} G'_0/P'_0 &= \text{space of lines in } \Psi^{n+1} \text{ in real } P^{n+2} \\ &= \text{space of pencils of mutually tangent oriented spheres in real } E^n \\ &= \text{space of oriented co-directions in real } E^n, \end{aligned}$$

a real form of  $G/P$ ; cf. 4.7.

Since  $G' = T^{-1} G T$ , we can exhibit  $G'_0/P'_0$ , as well as  $G_0/P_0$ , as a real form of the complex contact manifold  $G/P$ .  $T G'_0 P^{-1}$  is the real form of  $G = T G' T^{-1}$  consisting of the elements fixed under the conjugation obtained by transporting the conjugation  $g' \rightarrow \bar{g}'$  of  $G'$  to  $G$ , namely

$$g \rightarrow T \overline{(T^{-1} g T)} T^{-1} = S^{-1} \bar{g} S$$

where  $S = \bar{T} T^{-1}$ . In the case of  $n$  odd,

$$S = \left[ \begin{array}{cc|cc} 0 & 0 & 1_{l-2} & 0 \\ 0 & 1_2 & 0 & 0 \\ \hline 1_{l-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \end{array} \right];$$

in the case of  $n$  even,  $S$  has an additional initial row and column with a 1 in their common first entry and 0s elsewhere.  $S A S = A$  and  $S^2 = 1_{n+3}$ , so the complex conjugation  $\xi \rightarrow S^{-1} \bar{\xi}$  preserves the quadric  $\Omega^{n+1}$ . A point or line of  $\Omega^{n+1}$  is fixed under this conjugation exactly if it is the image under  $T$  of a real point or line of  $\Psi^{n+1}$ . The latter constitute the orbit on  $\Omega^{n+1}$  of  $T G'_0 T^{-1}$ . The isotropy subgroup in  $T G'_0 T^{-1}$  of the line  $l_0$  of  $\Omega^{n+1}$  is  $T G'_0 T^{-1} \cap P = T P'_0 T^{-1}$ . Furthermore, the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  in 5.1 is stable under the conjugation  $X \rightarrow S^{-1} \bar{X} S$  of  $\mathfrak{g}$ ; in fact, for

$$H = \text{diag} (h_1, \dots, h_l \mid -h_1, \dots, -h_l)$$

in  $\mathfrak{h}$ , we have

$$S^{-1} \bar{H} S = \text{diag} (-\bar{h}_1, \dots, -\bar{h}_{l-2}, \bar{h}_{l-1}, \bar{h}_l \mid \bar{h}_1, \dots, \bar{h}_{l-2}, -\bar{h}_{l-1}, -\bar{h}_l),$$

in case of  $n$  odd; the maximal root  $\rho = -\delta_{l-1} - \delta_l$  is real,  $\overline{\rho(S^{-1}\bar{H}S)} = \rho(H)$ . Hence, the contact structure on  $TG'_0T^{-1}/TP'_0T^{-1}$  is that obtained from  $G/P$  by 2.11. We conclude:

$G_0/P_0$  and  $TG'_0T^{-1}/TP'_0T^{-1}$ , the latter isomorphic to  $G'_0/P'_0$ , are two real forms of the complex contact manifold  $G/P$ .

5.6. We observed in 5.3 that the space of co-directions in complex projective space  $P^3$ , by means of Plücker's line geometry, is isomorphic to the space of lines in the quadric  $\Omega^4$  in complex  $P^5$ , and that this isomorphism makes real line geometry correspond to a real form of  $\Omega^4$ . We found in 5.4 and 5.5 that the space of oriented co-directions in complex Euclidean space  $E^3$  of Lie's higher sphere geometry, which is the space of lines in the quadric  $\Psi^4$  in complex  $P^5$ , is isomorphic to the space of lines in the quadric  $\Omega^4$  also, and that this isomorphism makes real sphere geometry correspond to a second real form of  $\Omega^4$ . That is, real line geometry and real sphere geometry are two distinct real forms of complex line geometry. The line-sphere transformation establishes the isomorphism of the spaces of lines in  $\Psi^4$  and lines in  $\Omega^4$ . The former places real sphere geometry in the foreground, the latter, real line geometry.

5.7. The isomorphism of 5.3 may be used to describe sphere geometry in terms of co-directions in complex  $P^3$ . Real sphere geometry then leads to the real form  $PSU(2,2)$  of  $PSL(4; \mathbb{C})$ .

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