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**Autor:** Fillmore, Jay P.

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in case of  $n$  odd; the maximal root  $\rho = -\delta_{l-1} - \delta_l$  is real,  $\overline{\rho(S^{-1}\bar{H}S)} = \rho(H)$ . Hence, the contact structure on  $TG'_0T^{-1}/TP'_0T^{-1}$  is that obtained from  $G/P$  by 2.11. We conclude:

$G_0/P_0$  and  $TG'_0T^{-1}/TP'_0T^{-1}$ , the latter isomorphic to  $G'_0/P'_0$ , are two real forms of the complex contact manifold  $G/P$ .

5.6. We observed in 5.3 that the space of co-directions in complex projective space  $P^3$ , by means of Plücker's line geometry, is isomorphic to the space of lines in the quadric  $\Omega^4$  in complex  $P^5$ , and that this isomorphism makes real line geometry correspond to a real form of  $\Omega^4$ . We found in 5.4 and 5.5 that the space of oriented co-directions in complex Euclidean space  $E^3$  of Lie's higher sphere geometry, which is the space of lines in the quadric  $\Psi^4$  in complex  $P^5$ , is isomorphic to the space of lines in the quadric  $\Omega^4$  also, and that this isomorphism makes real sphere geometry correspond to a second real form of  $\Omega^4$ . That is, real line geometry and real sphere geometry are two distinct real forms of complex line geometry. The line-sphere transformation establishes the isomorphism of the spaces of lines in  $\Psi^4$  and lines in  $\Omega^4$ . The former places real sphere geometry in the foreground, the latter, real line geometry.

5.7. The isomorphism of 5.3 may be used to describe sphere geometry in terms of co-directions in complex  $P^3$ . Real sphere geometry then leads to the real form  $PSU(2,2)$  of  $PSL(4; \mathbb{C})$ .

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Jay P. Fillmore

Department of Mathematics  
University of California at San Diego  
La Jolla, California 92093