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with $m \equiv 2 \pmod 4$. The *Picard-Lefshetz automorphisms* T_i of $H_m(F)$ for $i = 1, ..., \mu$ are defined by

$$T_i(x) = x + (\delta_i, x)\,\delta_i\,.$$

Another way of writing T_i is

$$T_i(x) = x - 2\,\frac{(\delta_i, x)}{(\delta_i, \delta_i)}\,\delta_i$$

which shows that T_i is a reflection in δ_i [Lamotke].

The *monodromy group* of f is the subgroup of the automorphism group of $H_m(F)$ generated by $T_1, ..., T_\mu$. This group depends only on f, since it may also be defined as a representation of the *braid group* of f, which is the fundamental group of the complement of the bifurcation diagram in the base space of the versal unfolding of f [Arnold 3, §2.8]. (Here is a direct proof that the monodromy group of f is independent of the choice of nearby Morse function $\bar{f}$ and paths $\alpha_1, ..., \alpha_\mu$: The set $D_\delta^2 - \{t_1, ..., t_\mu\}$ is the base space of a fiber bundle with fiber F, so $\pi_1(D_\delta^2 - \{t_1, ..., t_\mu\}, \delta)$ acts on $H_m(F)$. The image of β_i in Aut $H_m(F)$ is T_i; since $\beta_1, ..., \beta_\mu$ generate π_1, the monodromy group is the image of π_1 in Aut $H_m(F)$. Thus the monodromy group is independent of the choice of $\alpha_1, ..., \alpha_\mu$. It is independent of the choice of $\bar{f}$ since any two nearby Morse functions with μ distinct critical values can be joined by a family of such functions.)

Characterization B8. The monodromy group of f is finite.

Characterization B5 implies Characterization B8 since the automorphism group of any positive definite integral lattice is finite. In fact, the monodromy groups are precisely the Coxeter groups of the corresponding quadratic form diagram. Conversely, [Gabrielov 3] shows that if a germ f topologically degenerates to a germ g, then the monodromy group of f is a quotient of a subgroup of the monodromy group of g. Since the monodromy groups of the germs in Table 2b are infinite [Gabrielov 1], Proposition 10.1 shows that Characterization B8 implies Characterization B1.

16. Weighted homogeneous polynomials

A polynomial $g(z_0, ..., z_n)$ is *weighted homogeneous* if there are positive rational numbers $w_0, ..., w_n$ (the *weights*) such that $g(z_0, ..., z_n)$ may be written as a sum of monomials $z_0^{i_0} \dots z_n^{i_n}$ with $i_0/w_0 + ... + i_n/w_n = 1$

[Milnor 1, p. 75; Orlik and Wagreich]. Another way of saying this is that if the variables z_i are weighted by $1/w_i$, then g is homogeneous of degree one, that is, $g(\lambda^{1/w_0}z_0, ..., \lambda^{1/w_n}z_n) = \lambda g(z_0, ..., z_n)$ for all complex numbers λ. All the germs in Table 1 are weighted homogeneous with weights as listed in Column 7. These germs remain weighted homogeneous upon adding squares of new variables, each weighted by 2. It is proved in [Saito 1, Lemma 4.3] that the weights of a germ g are uniquely determined (up to permutation) by the analytic isomorphism type of $g^{-1}(0)$.

Characterization B9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + ... + w_n^{-1} > n/2$.

The equivalence of Characterizations B2 and B9 is proved in [Saito 2, Satz 2.11]. (The r there is $w_0^{-1} + ... + w_n^{-1}$.)

Appendix I

Nine Characterizations of Almost-Simple Critical Points (Simple Elliptic Singularities)

Almost-simple critical points can also be characterized in several ways. The nine characterizations presented in this appendix are analogues of some of those in the main text.

THEOREM C. *Let $f(z_0, ..., z_n)$ with $n \geqslant 2$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that $\mathbf{0}$ is an isolated critical point. Then Characterizations C1 through C9 are equivalent.*

Characterization C1. The germ f is right equivalent to one of the germs in Table 2b.

Characterization C2. The germ f is contact equivalent to one of the germs in Table 2b.

The equivalence of these characterizations follows from Proposition 9.1.