§2. Weil-Milgram quadratic reciprocity law

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **26 (1980)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

§ 2. Weil-Milgram quadratic reciprocity law

Let A_F denote the adele group of the field F i.e. the group of infinite vectors $(\dots x_v \dots)$ where v runs through all valuations of F, x_v is an element of completion F_v, and all x_v except for finite number of v's are integers. F is diagonally embedded in A_F. Let χ denote a character of A_F trivial on F. Let χ_0 be the following special character of such type. On the archimedian component v of the adele $\{x_v\} \in A_F$, χ_0 takes the value $\exp(-2\pi i \mathrm{Tr} x_v)$, and on a non-archimedian component $\exp(2\pi i \mathrm{Tr} x_v)$. Here Tr denotes the absolute trace from the v-component F_v of A_F to $\mathbf{Q}_{\bar{v}}$ where $\bar{v}$ is a valuation of $\mathbf{Q}$ over which lies v. Recall that exponent of p-adic number a is defined as exponent of a component a_1 in presentation $a = a_1 + a_2$ where $a_1 \in \mathbf{Q}$, $a_1 = p^{-n} a_1'$, $a_1' \in \mathbf{Z}$, and $a_2 \in \mathbf{Z}_p$. Any character χ of the type above has the form $x \to \chi_0(ax)$ for some rational a.

Let q be a quadratic form defined on the vector space V over F_v where v is one of the non-archimedian valuations of F. Suppose that L is a lattice in V such that $\chi(q(x)) = 1$ for any $x \in V$. The dual lattice $L^{\#}$ is defined as follows

$$L^{\#} = \{ h \in V \mid \chi(\tilde{q}(x, h)) = 1 \quad \text{for} \quad \forall x \in V \} \tag{6}$$

where

$$\tilde{q}(x, y) = q(x+y) - q(x) - q(y) \tag{7}$$

is the bilinear form associated to q. Then the correspondence $q \mapsto \gamma_v^\chi(q)$ where

$$\gamma_v^\chi(q) = \sum_{h \in L^{\#}/L} \chi(q(h)) \Big/ \Big| \sum_{h \in L^{\#}/L} \chi(q(h)) \Big| \tag{8}$$

defines a character of the Witt group $W(F_v)$ ([5]). (Over a field of zero characteristic we can identify the Witt group of quadratic forms with the Witt group of bilinear forms by the correspondence (7)). For an archimedian valuation v, the character γ_v^χ is defined as follows.

$$\gamma_v^\chi(q) = \exp \frac{-\pi i \sigma(q)}{4} \tag{9}$$

if F_v is $\mathbf{R}$, and

$$\gamma_v^\chi(q) = 1 \tag{10}$$

if F_v is **C**. ($\sigma(q)$ denotes the signature of the quadratic form q). Now suppose that q is a quadratic form over the field F. Then q defines quadratic forms q_v over all F_v and the Weil quadratic reciprocity law asserts that

$$\prod_v \gamma_v^\chi(q_v) = 1 \tag{11}$$

where v runs through all valuations of F.

If S is a symmetric bilinear form over **Z**, on the lattice L such that $q(x) = S(x, x)$ is an even quadratic form, then applying (11) to $\varphi(x) = \frac{1}{2} q(x)$ and the character χ_0 of the ring $A_{\mathbf{Q}}$ defined above, one concludes that

$$e^{\frac{\pi i \sigma(q)}{4}} = \sum_{x \in L_p^{\#} / L_p} e^{2\pi i \varphi(x)} \Big/ \Big| \sum_{x \in L_p^{\#} / L_p} e^{2\pi i \varphi(x)} \Big| \tag{12}$$

where L_p is the lattice of integer vectors in the p-adic completion of L. This is the essential part of Milgram's formula ([4]).

Now let us consider properties of the character γ^χ in more detail.

Let $F_{\mathfrak{p}}$ be one of the completions of F where $\mathfrak{p}$ is a non-dyadic prime ideal.

LEMMA 1. Let q be a quadratic form over $F_{\mathfrak{p}}$. Let a be a unit in $F_{\mathfrak{p}}$. Denote by (aq) the quadratic form defined by $(aq)(x) = a \,.\, q(x)$.
Then

$$\gamma_{\mathfrak{p}}^\chi(aq) = \left(\frac{a}{(\det \tilde{q}) \,.\, \mathfrak{s}^{rkq}}\right) \gamma_{\mathfrak{p}}^\chi(q) \tag{13}$$

where $(-)$ is the quadratic residue symbol, $\mathfrak{s}$ is the support of the character χ, and rkq is rank of the form q.

Remark. $\det \tilde{q}$ is defined up to a square in $F_{\mathfrak{p}}$, and therefore the quadratic residue symbol in (13) is well defined.

Now we consider dyadic valuations.

LEMMA 2. Let q denote a quadratic form over a ring of integers $R_{\mathfrak{p}}$ of the dyadic field $F_{\mathfrak{p}}$ such that the determinant of the associated form $\tilde{q}$ (see (7)) is a unit in $R_{\mathfrak{p}}$. Let χ be a character of $F_{\mathfrak{p}}$ with support $R_{\mathfrak{p}}$. If $\mathfrak{p}$ is tamely ramified over **Q** then

$$\operatorname{Arf}(q \bmod \mathfrak{p}) = \gamma_{\mathfrak{p}}^\chi(2q)\,. \tag{14}$$

Otherwise

(15) $$\gamma_{\mathfrak{p}}^{\chi}(2q) = 1\,.$$

Remark. The condition on $\det \tilde{q}$ implies non-degeneracy of q at $\mathfrak{p}$.

§ 3. Proof of the main theorem

Note that the rank of q is even because determinant of the associated bilinear form is odd. Therefore

(16) $$\gamma_v^{\chi}(aq) = \left(\frac{a}{(\det \tilde{q})}\right) \gamma_v^{\chi}(q)$$

for *any* character χ.

Now let us apply the Weil reciprocity law for the character χ with support in dyadic components equal to the integers in the corresponding ring, and to the forms q and $2q$.

We have

$$\prod_v \gamma_v^{\chi}(2q) = 1$$

$$\prod_v \gamma_v^{\chi}(q) = 1\,.$$

For an archimedian components we have $\gamma_v^{\chi}(2q) = \gamma_v^{\chi}(q)$ because both depend only on the signatures. Therefore dividing those two identities, and using lemma 2 and (16) we obtain the identity (4).

Remark. Levine's lemma which in a specialization of the theorem for $R = \mathbf{Z}$ in fact follows from Milgram's formula (12). We should not worry about ramification. Therefore lemma 1 can be used for the character χ_0 and is actually a classical property of Gauss sums ([2]). Lemma 2 in this case essentially contains in [1].

§ 4. Proof of the lemmas

Proof of lemma 1. The Witt group of quadratic forms over a field of zero characteristic is generated by one-dimensional forms ([4]). Because γ^{χ} is a character of the Witt group it is enough to check the lemma for forms of one variable. Let π be a local parameter. Suppose that $q(x) = \alpha\, \pi^b\, x^2$,