2. The Mazur-Gelfand theorem

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A *normed algebra* is an associative linear algebra A over the real or complex field which is also a normed linear space satisfying $\| xy \| \leqslant \| x \| \cdot \| y \|$ for every x and y in A. If A is complete in this norm, it is called a *Banach algebra.*

In 1938 Stanislaw Mazur [57] announced the following classification theorem for real normed division algebras:

Theorem 2.1. [Mazur]. *A real normed algebra with identity in which every nonzero element has an inverse is isomorphic to either* **R**, **C**, *or the quaternions* **H**.

An immediate consequence of this result, which classifies normed division algebras over **C**, is known as the Mazur-Gelfand theorem:

Theorem 2.2. [Mazur-Gelfand]. *A complex normed algebra with identity in which every nonzero element has an inverse is isomorphic to the complex numbers.*

This complex version follows in a standard way from Theorem 2.1 since every complex normed algebra is also a real normed algebra, and the possibilities of **R** and **H** are easily eliminated in the complex case.

An historical precursor to Mazur's theorem was published by Alexander Ostrowski in 1918 [65]. It states that every field with an archimedean valuation is topologically isomorphic with a subfield of **C** carrying the ordinary absolute value as its valuation. If the field has additionally the structure of a real vector space, then the possibilities are further reduced to **R** or **C**.

The details of Mazur's proof were too lengthy to be included in his announcement, and it was Gelfand who furnished the first published proof [38] of the complex version, which bears his name. His proof, different from Mazur's, uses a generalized form of Liouville's theorem from complex analysis. The theorem was established independently by Lorch [55] whose proof likewise was based on Liouville's theorem; he points out that substantially the same argument was given earlier by Taylor [91]. We now record this elegant proof in a form which uses the classical version of Liouville's theorem.

Gelfand's proof of the Mazur-Gelfand Theorem 2.2. For any element x of the complex normed algebra A with identity e, we show that $x = \lambda e$

for some complex λ. Suppose to the contrary that $x - \lambda e \neq 0$ for all λ in **C**. Since A is a division algebra, it follows that $x - \lambda e$ is invertible for all λ, i.e., $(x-\lambda e)^{-1}$ exists. Let $x(\lambda) = (x-\lambda e)^{-1}$. By the Hahn-Banach theorem there is a bounded linear functional L on A such that $L(x^{-1}) = 1$. Define $g : \mathbf{C} \to \mathbf{C}$ by $g(\lambda) = L(x(\lambda))$; then $g(0) = 1$. Moreover, g is an entire function. Indeed, since $x(\lambda) - x(\mu) = (\lambda-\mu)\, x(\lambda)\, x(\mu)$ for λ, μ in **C**, it follows that

$$\lim_{\lambda \to \mu} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} = \lim_{\lambda \to \mu} L(x(\lambda)\, x(\mu)) = L(x(\mu)^2).$$

Further $|g(\lambda)| \leqslant \|L\| \, \|x(\lambda)\|$ and since $x(\lambda) \to 0$ as $|\lambda| \to \infty$, $g(\lambda) \to 0$. By Liouville's theorem the bounded entire function g is constant; hence $g \equiv 0$. This is a contradiction since $g(0) = 1$, and the proof is complete.

The *spectrum* of an element x of a complex algebra with identity e is the set $\sigma(x) = \{\lambda \in \mathbf{C} : x - \lambda e \text{ is singular}\}$, so Gelfand's proof can be viewed as a demonstration that the spectrum of any element of a complex normed algebra with identity is nonempty. This fact together with the application of Liouville's theorem forms a continuous thread running through the generalizations and related results presented in this paper.

3. Classification of real normed division algebras

Although it does not appear to be widely known, Mazur's original paper on normed division algebras [57] considers *only* the case of algebras over **R**. If a real division algebra is also finite-dimensional, the classical theorem of Frobenius classifies it as **R**, **C** ,or **H**. Mazur demonstrated finite-dimensionality in two steps: first he used a rather lengthy argument involving analytic function theory to show that it cannot contain a subalgebra isomorphic to the rational functions in one indeterminate with real coefficients. He then quoted an algebraic theorem to the effect that every real infinite-dimensional division algebra must contain such a subalgebra. The details of the first step may now be found in W. Żelazko's book [109, pp. 18-22].

F. F. Bonsall and J. Duncan [30] have given a more direct and self-contained proof of Mazur's theorem, which relies on precisely the same analytic fact as Gelfand's proof of the complex version; namely that every element of a complex normed algebra with identity has nonempty spectrum. They modify a standard proof of Frobenius' theorem (*vid.* Pontrjagin