

## 2. The Mazur-Gelfand theorem

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## 2. THE MAZUR-GELFAND THEOREM

A *normed algebra* is an associative linear algebra  $A$  over the real or complex field which is also a normed linear space satisfying  $\|xy\| \leq \|x\| \cdot \|y\|$  for every  $x$  and  $y$  in  $A$ . If  $A$  is complete in this norm, it is called a *Banach algebra*.

In 1938 Stanislaw Mazur [57] announced the following classification theorem for real normed division algebras:

**THEOREM 2.1.** [Mazur]. *A real normed algebra with identity in which every nonzero element has an inverse is isomorphic to either  $\mathbf{R}$ ,  $\mathbf{C}$ , or the quaternions  $\mathbf{H}$ .*

An immediate consequence of this result, which classifies normed division algebras over  $\mathbf{C}$ , is known as the Mazur-Gelfand theorem:

**THEOREM 2.2.** [Mazur-Gelfand]. *A complex normed algebra with identity in which every nonzero element has an inverse is isomorphic to the complex numbers.*

This complex version follows in a standard way from Theorem 2.1 since every complex normed algebra is also a real normed algebra, and the possibilities of  $\mathbf{R}$  and  $\mathbf{H}$  are easily eliminated in the complex case.

An historical precursor to Mazur's theorem was published by Alexander Ostrowski in 1918 [65]. It states that every field with an archimedean valuation is topologically isomorphic with a subfield of  $\mathbf{C}$  carrying the ordinary absolute value as its valuation. If the field has additionally the structure of a real vector space, then the possibilities are further reduced to  $\mathbf{R}$  or  $\mathbf{C}$ .

The details of Mazur's proof were too lengthy to be included in his announcement, and it was Gelfand who furnished the first published proof [38] of the complex version, which bears his name. His proof, different from Mazur's, uses a generalized form of Liouville's theorem from complex analysis. The theorem was established independently by Lorch [55] whose proof likewise was based on Liouville's theorem; he points out that substantially the same argument was given earlier by Taylor [91]. We now record this elegant proof in a form which uses the classical version of Liouville's theorem.

*Gelfand's proof of the Mazur-Gelfand Theorem 2.2.* For any element  $x$  of the complex normed algebra  $A$  with identity  $e$ , we show that  $x = \lambda e$

for some complex  $\lambda$ . Suppose to the contrary that  $x - \lambda e \neq 0$  for all  $\lambda$  in  $\mathbf{C}$ . Since  $A$  is a division algebra, it follows that  $x - \lambda e$  is invertible for all  $\lambda$ , i.e.,  $(x - \lambda e)^{-1}$  exists. Let  $x(\lambda) = (x - \lambda e)^{-1}$ . By the Hahn-Banach theorem there is a bounded linear functional  $L$  on  $A$  such that  $L(x^{-1}) = 1$ . Define  $g : \mathbf{C} \rightarrow \mathbf{C}$  by  $g(\lambda) = L(x(\lambda))$ ; then  $g(0) = 1$ . Moreover,  $g$  is an entire function. Indeed, since  $x(\lambda) - x(\mu) = (\lambda - \mu)x(\lambda)x(\mu)$  for  $\lambda, \mu$  in  $\mathbf{C}$ , it follows that

$$\lim_{\lambda \rightarrow \mu} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} = \lim_{\lambda \rightarrow \mu} L(x(\lambda)x(\mu)) = L(x(\mu)^2).$$

Further  $|g(\lambda)| \leq \|L\| \|x(\lambda)\|$  and since  $x(\lambda) \rightarrow 0$  as  $|\lambda| \rightarrow \infty$ ,  $g(\lambda) \rightarrow 0$ . By Liouville's theorem the bounded entire function  $g$  is constant; hence  $g \equiv 0$ . This is a contradiction since  $g(0) = 1$ , and the proof is complete.

The *spectrum* of an element  $x$  of a complex algebra with identity  $e$  is the set  $\sigma(x) = \{\lambda \in \mathbf{C} : x - \lambda e \text{ is singular}\}$ , so Gelfand's proof can be viewed as a demonstration that the spectrum of any element of a complex normed algebra with identity is nonempty. This fact together with the application of Liouville's theorem forms a continuous thread running through the generalizations and related results presented in this paper.

### 3. CLASSIFICATION OF REAL NORMED DIVISION ALGEBRAS

Although it does not appear to be widely known, Mazur's original paper on normed division algebras [57] considers *only* the case of algebras over  $\mathbf{R}$ . If a real division algebra is also finite-dimensional, the classical theorem of Frobenius classifies it as  $\mathbf{R}$ ,  $\mathbf{C}$ , or  $\mathbf{H}$ . Mazur demonstrated finite-dimensionality in two steps: first he used a rather lengthy argument involving analytic function theory to show that it cannot contain a subalgebra isomorphic to the rational functions in one indeterminate with real coefficients. He then quoted an algebraic theorem to the effect that every real infinite-dimensional division algebra must contain such a subalgebra. The details of the first step may now be found in W. Żelazko's book [109, pp. 18-22].

F. F. Bonsall and J. Duncan [30] have given a more direct and self-contained proof of Mazur's theorem, which relies on precisely the same analytic fact as Gelfand's proof of the complex version; namely that every element of a complex normed algebra with identity has nonempty spectrum. They modify a standard proof of Frobenius' theorem (*vid.* Pontrjagin