7. FURTHER GENERALIZATIONS AND RELATED RESULTS

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$\beta \geqslant 1$). Then $\lambda - x$ is invertible and $\lambda - (x+y) = (\lambda - x)\,[1-(\lambda-x)^{-1}y]$ while $|(\lambda-x)^{-1}y|_\sigma \leqslant \beta\,|(\lambda-x)^{-1}|_\sigma\,|y|_\sigma \leqslant \beta\,(|\lambda|-|x|_\sigma)^{-1}\,|y|_\sigma < 1$. Thus $\lambda - (x+y)$ is invertible and the conclusion follows with $\alpha = \max\{\beta, 1\}$. If A has no identity, again assume $\beta \geqslant 1$ and that $|\lambda| > \beta\,|x|_\sigma + \beta\,|y|_\sigma$. Since $\lambda - x$ is invertible in A_1, we have $(\lambda-x)^{-1} = v + u$ where $v \in A$ and $\mu \in \mathbf{C}$. From $(\lambda - x)(\mu + v) = 1$ we have $\mu = 1/\lambda$, and hence $v = (\lambda-x)^{-1} - 1/\lambda$. Now $\lambda - (x+y) = (\lambda-x)\,[1-(\lambda-x)^{-1}y] = (\lambda-x)\,[1-(1/\lambda y - vy] = (\lambda-x)\,[1-vy\,(1-(1/\lambda)y)^{-1}]\,(1-(1/\lambda)y)$, where $1 - (1/\lambda)y$ is invertible since $|\lambda| > |y|_\sigma$. Since A is an ideal in A_1, we have $|vy\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant \beta\,|v|_\sigma\,|y\,(1-(1/\lambda)y)^{-1}|_\sigma$. But

$$|v|_\sigma \leqslant \frac{|x|_\sigma}{|\lambda|\,(|\lambda| - |x|_\sigma)} \leqslant \frac{|x|_\sigma}{|\lambda|\,(|\lambda| - \beta|x_{:\sigma})}$$

and $|y\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant |\lambda| \cdot |y|_\sigma/(|\lambda|-|y|_\sigma) \leqslant |\lambda| \cdot |y|_\sigma/(|\lambda|-\beta|y|_\sigma)$. Multiplying these estimates we obtain $|vy\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant \beta\,|x|_\sigma\,|y|_\sigma/(|\lambda|-\beta|x|_\sigma)\,(|\lambda|-\beta|y|_\sigma) < 1$. So again we may take $\alpha = \max\{\beta, 1\}$.

Proposition 6.6. *If $|\cdot|_\sigma$ is subadditive on A, then A is almost commutative.*

Proof. Since $|\cdot|_\sigma$ must be submultiplicative, Corollary 5.5 (to the results of Hirschfeld-Żelazko) states that $A/\mathrm{Rad}\,(A)$ must be commutative.

It is of interest that the uniform continuity of $|\cdot|_\sigma$ implies its Lipschitz continuity. It is easy to see that the Lipschitz constant can be taken as $1/\varepsilon$ where $\|x - y\| < \varepsilon$ implies $\|x|_\sigma - |y|_\sigma| \leqslant 1$. We have already seen that the subadditivity of $|\cdot|_\sigma$ implies its Lipschitz continuity.

7. Further generalizations and related results

During the past forty years the general subject of this paper has received attention from many authors. Our purpose here is to give a brief discussion of some of the relevant literature.

Ramaswami studies in [71] the Mazur-Gelfand theorem under minimal hypothesis. He weakens the associative law and also the triangle inequality; the former in several ways. In all he gives six different sets of sufficient conditions for a "generalized" complex Banach algebra to coincide with the complex field. He treats real Banach algebras in the same spirit.

Elementary proofs of the Mazur-Gelfand theorem which avoid direct appeal to complex function theory (in particular to Liouville's theorem) have been given by Kametani [51], Ono [64], Ramaswami [71], Rickart [72], Stone [84], and Tornheim [93]. Stone's proof is based on completeness of the algebra and a study of the behavior of the sequence $x^n (x^n - r^n)$ as $n \to \infty$, $r \geqslant 0$, where x is an element which is not a complex multiple of the identity. Tornheim [93] observes that if A is a normed field, $y \in A$, $y \notin \mathbf{C}$, then the function $f : \mathbf{C} \to \mathbf{R}$ defined by

$$f(\lambda) = || 1/(y - \lambda e) ||$$

is continuous, positive and small for large λ. Hence f takes on a maximum value $M > 0$ on a closed, bounded subset of $\mathbf{C}$. He then gives an elementary argument (that does not depend on completeness) which shows that this is impossible. The arguments of Ono [64] and Rickart [72] were obtained independently and nearly simultaneously; both are based on properties of roots of unity. Rickart's proof appears in his well known book [73].

Various generalizations of the Mazur-Gelfand theorem and Theorem 3.1 to topological algebras with identity have been considered by Allan [5], Arens [6], Aurora [17], Edwards [37], Ramaswami [71], Shafarevich [76], Shilov [77], Stone [84], Turpin [94], and Żelazko [101-108]. For the most part these papers are concerned with either omitting the assumption of a norm altogether or else significantly weakening the properties of the norm. For example, Arens [6] shows that a convex topological linear division algebra over $\mathbf{C}$ with continuous inversion is isomorphic to $\mathbf{C}$. On the other hand, he observes that the algebra of holomorphic functions on an open subset of $\mathbf{C}$ with the usual topology is a non-normable, convex, metrizable algebra which has no nonzero topological divisors of zero but still is not a division algebra.

The extensive studies of Aurora [17-25] relating to the Mazur-Gelfand theorem and the general classification of topological fields are quite significant. Generally speaking, he centers his attention on associative rings with identity which are furnished with a norm satisfying a wide variety of hypotheses. His results are numerous, and, among other things, contain many of the previously discussed results as special cases. For more information the reader may consult the individual papers.

A very readable account of Żelazko's (and others) work up to 1964 can be found in his Yale lecture notes [104] (see also [105]). His more recent work [106, 107, 108] is concerned with extending known properties of topological divisors of zero. For example, in [106] a topological ring is said to have *generalized topological divisors of zero* if there exists at least one pair of subsets P, Q of A such that 0 is in the closure of PQ, but is not in the closure of P or Q. A locally convex topological algebra A over the complex numbers is called *m-convex* if its topology is defined by a family of semi-norms for which $\| xy \| \leqslant \| x \| \; \| y \|$ for all x, y in A. The main theorem in [106] states that an m-convex topological algebra which is not the complex numbers always has generalized topological divisors of zero. In [107] he shows that a *real* m-convex algebra must have generalized topological divisors of zero or be homeomorphically isomorphic to either the reals, complexes, or quaternions. In [108] Kaplansky's theorem (4.2) is extended to real p-normed algebras. (Roughly, a *p-normed algebra* is an algebra with a complete algebra norm with the usual homogeneity replaced by $\| \lambda x \| = | \lambda |^p \| x \|$, $0 < p \leqslant 1$.)

The subject of absolute-valued algebras has been studied by Albert [2, 3, 4], Gleichgewicht [41], Strzelecki [85, 86, 87], Urbanik [95], Urbanik and Wright [96], and Wright [99]. In these papers it is not assumed that the algebra is associative. In particular, an example provided in [96] shows that Theorem 3.1 fails for non-associative algebras. In this more general context it is proved by Urbanik and Wright [96] that a real, absolute-valued algebra with identity is isomorphic to either the reals, the complexes, the quaternions, or the Cayley-Dickson algebra. Their proof consists in showing that the algebra in question must be algebraic (i.e., the algebra $A(x)$ generated by x is finite dimensional for each x); then the theorem follows from one of Albert's theorems (See [3]).

Two papers on real Hilbert algebras deserve mention. A *Hilbert algebra* is a Hilbert space which is also a Banach algebra relative to the inner product norm $\| x \| = (x|x)^{1/2}$. In the real case Ingelstam proved that if such an algebra has an identity of unit norm, it must be the reals, complexes or quaternions. His argument has been considerably simplified by Smiley [81]. In the complex case Hirschfeld [45] considers *Hilbertizable* algebras, i.e., Banach algebras which support a comparable norm which makes the algebra a Hilbert space, but is not necessarily submultiplicative. Using ideas of Ingelstam and Smiley he shows that a complex Hilbertizable algebra which is semi-simple and has an identity of unit (Hilbert) norm must be the complex numbers. He remarks that van Castern using a different line of argument has shown the assumption of semi-simplicity to be superfluous.

Jan van Castern's similar result, also giving necessary and sufficient conditions for a complex algebra which is a normed linear space to be the complex numbers, is as follows [32]: Let A be an algebra over **C** with identity e. Suppose that A as a vector space is normed so that $\| e \| = 1$

and the unit ball is smooth at e (i.e., $\{\phi \in (A, \|\cdot\|)' : \|\phi\| = \phi(e) = 1\}$ is a singleton). Then $A = \mathbf{C}e$ if and only if there is a norm $\|\cdot\|_1$ on A for which $\|(x+e)^n\|_1 \leqslant \exp(n\|x\|)$ for all $n \in \mathbf{N}$ and $x \in A$.

A characterization of commutativity in C^*-algebras is given by Crabb-Duncan-McGregor [34] again in terms of a norm inequality: A C^*-algebra A is commutative if and only if

$$\|x + y\| \leqslant 1 + \|xy\|$$

for all hermitian elements $x, y \in A$ with $\|x\| = \|y\| = 1$. Some generalizations of this can be found in Duncan and Taylor [35].

A version of Edward's theorem (4.13) using the spectral radius in place of the norm has been established by Aupetit. The statement and proof given in [12] contain a gap which will be corrected in Aupetit's forthcoming monograph [16]. The correct statement is: If A is a real Banach algebra with identity containing a nonempty open set U of invertible elements satisfying $\rho(x)\rho(x^{-1}) = 1$, then $A/\mathrm{Rad}\,A$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, or $M_2(\mathbf{R})$. If A is complex, $A/\mathrm{Rad}\,A$ is isomorphic to $\mathbf{C}$.

There are in addition two complex versions of this theorem in the context of *-algebras:

Let A be a complex Banach *-algebra with identity the set of whose hermitian elements contain a nonempty open set U of invertibles such that for every $x \in U$, $\rho(x)\rho(x^{-1}) = 1$. Then $A/\mathrm{Rad}\,A$ is isomorphic to $\mathbf{C}$ with the involution $u \to \bar{u}$ or to $\mathbf{C}^2$ with the involution $(u, v) \to (\bar{u}, \bar{v})$. The other version has precisely the same statement except that ρ is replaced by $\|\cdot\|$.

Aupetit has also obtained results along the lines of the Hirschfeld-Żelazko theorems in the context of *-algebras. We cite the following two theorems from [10]:

(1) Let A be a Banach *-algebra and $k > 0$ such that for all hermitian h one has $\rho(h) \geqslant k\|h\|$ and $\sigma(h)$ has no interior points and a finite number of holes. Then if A has no nilpotents, A is commutative.

(2) Let A be a Banach *-algebra and $k > 0$ such that for every hermitian h one has $\rho(h) \geqslant k\|h\|$ and $\sigma(h)$ has no interior points and a finite number of holes. Then if A has no non-nilpotent quasi-nilpotents, all its irreducible representations are finite dimensional.

A generalization of commutativity called P-commutativity has been studied by W. Tiller in [92]. The definition is as follows: For each positive functional f on a complex *-algebra A, let $I_f = \{x \in A : f(x^*x) = 0\}$

and $P = \cap I_f$, where the intersection is taken over all positive functionals on A. The algebra A is called *P-commutative* if $xy - yx \in P$ for all x, y in A. Tiller establishes the following two theorems relating properties of the spectral radius to P-commutativity:

(1) Let A be a Banach *-algebra which is symmetric and P-commutative. Then if $x, y \in A$, $\rho(xy) \leqslant \rho(x)\rho(y)$ and $\rho(x+y) \leqslant \rho(x) + \rho(y)$.

(2) Let A be a Banach *-algebra with bounded approximate identity. If $\rho(x^*x) \leqslant \rho(x)^2$ for every x in A, then A is P-commutative.

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