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APPENDIX:
TORSION POINTS OF ABELIAN VARIETIES
IN CYCLOTOMIC EXTENSIONS

by Kenneth A. RIBET ¹⁾

Let k be a number field, and let \bar{k} be an algebraic closure for k . For each prime p , let K_p be the subfield of k obtained by adjoining to k all p -power roots of unity in \bar{k} . Let K be the compositum of all of the K_p , i.e., the field obtained by adjoining to k all roots of unity in \bar{k} .

Suppose that A is an abelian variety over k . Mazur has raised the question of whether the groups $A(K_p)$ are finitely generated [4]. In this connection, H. Imai [1] and J.-P. Serre [5] proved (independently) that the *torsion subgroup* of $A(K_p)$ is finite for each p . The aim of this appendix is to prove that more precisely one has the following theorem, cf. [3], §II, Remark 3.

THEOREM 1. *The torsion subgroup $A(K)_{\text{tors}}$ of $A(K)$ is finite.*

Let G be the Galois group $\text{Gal}(\bar{k}/k)$ and let H be its subgroup $\text{Gal}(\bar{k}/K)$. For each positive integer n , let $A[n]$ be the kernel of multiplication by n in $A(\bar{k})$. For each prime p , let V_p be the \mathbf{Q}_p -adic Tate module attached to A . If M is one of these modules, we denote by M^H the set of elements of M left fixed by H . Since H is normal in G , M^H is stable under the action of G on M .

Because of the structure of the torsion subgroup of $A(\bar{k})$, one sees easily that Theorem 1 is equivalent to the conjunction of the following two statements:

THEOREM 2. *For all but finitely many primes p , we have $A[p]^H = 0$.*

THEOREM 3. *For each prime p , we have $V_p^H = 0$.*

Indeed, Theorem 2 asserts the vanishing of the p -primary part of $A(K)_{\text{tors}}$, while Theorem 3 asserts the finiteness of this p -primary part.

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In proving these statements, we visibly have the right to replace k by a finite extension of k . Therefore, using ([SGA 71], IX, 3.6) we can (and will) assume that A/k is semistable. Next, consider the largest subextension k' of K/k which is unramified at all finite places of k .

LEMMA. For each prime p , let L_p be the largest extension of k in K which is unramified at all places of k except for primes dividing p and the infinite places of k . Then L_p is the compositum $k'K_p$.

Proof. Let A be the Galois group $\text{Gal}(K/k)$, viewed as a subgroup of \hat{Z}^* . We consider \hat{Z}^* as the direct product of its two subgroups Z_p^* and $\prod_{l \neq p} Z_l^*$. Let I (resp. J) be the subgroup of A generated by the inertia groups of A for primes of k which divide p (resp. which do not divide p). Then I is a subgroup of Z_p^* , while J is a subgroup of $\prod_{l \neq p} Z_l^*$. The product $I \times J$ is the subgroup of A generated by all inertia groups of A . We have $J = \text{Gal}(\bar{k}/L_p)$, $I \times J = \text{Gal}(\bar{k}/k')$, and $\text{Gal}(\bar{k}/K_p) = A \cap \left(\prod_{l \neq p} Z_l^*\right)$. Now $\text{Gal}(\bar{k}/k'K_p)$ is the intersection of the two Galois groups $\text{Gal}(\bar{k}/k')$ and $\text{Gal}(\bar{k}/K_p)$. Putting these facts together, we prove the desired assertion.

We now replace k by its finite extension k' . With this replacement made, K_p becomes equal to L_p . Furthermore, for odd primes p , the largest extension of k in K which is unramified outside p and infinity and which has degree prime to p is the field obtained by adjoining to k the p -th roots of unity in \bar{k} .

Proof of Theorem 2. We shall consider only primes p which are odd, unramified in k , and such that A has good reduction at at least one prime of k dividing p . Let p be such a prime and v a prime of k over p at which A has good reduction. Suppose that the G -module $A[p]^H$ is non-zero, and let W be a simple G -submodule of this module. The algebra $\text{End}_G W$ is a finite field F , and the action of G on W is given by a character

$$\phi: G \rightarrow F^*$$

since the action of G on $A[p]^H$ is abelian. (Here the point is simply that G/H is an abelian group.) In particular, the image of G in $\text{Aut}(A[p])$ has order prime to p . On the other hand, the character ϕ is unramified at primes of k not dividing p because A/k is semistable. By the discussion following the lemma, we know that ϕ factors through the quotient $\text{Gal}(k(\mu_p)/k)$ of G ; here, μ_p denotes the group of p -th roots of unity. In particular, ϕ must have order dividing $p - 1$, so that its

values lie in the prime field F_p . Since W was chosen to be simple, its dimension over F_p must be 1; i.e., W is a group of order p .

Let $\chi: G \rightarrow F_p^*$ be the mod p cyclotomic character, i.e., the character giving the action of G on μ_p . Since ϕ factors through $\text{Gal}(k(\mu_p)/k)$, we may write ϕ in the form χ^n , where n is an integer mod $(p-1)$. We claim that n can only be 0 or 1.

To verify this claim, it is enough to check that it is true after we replace G by an inertia group I in G for the prime v , since χ is totally ramified at v . We remark that W is the I -module associated to a finite flat commutative group scheme \mathcal{W} over the ring of integers of the completion of k at v , since v is such that A has good reduction at v . Because \mathcal{W} has order p , the classification of Tate-Oort ([8], especially pp. 15-16) applies to \mathcal{W} . Because v is absolutely unramified, the classification shows immediately that \mathcal{W} is either étale or the dual of an étale group. In the former case, I acts trivially on W ; in the latter case, I acts on W via χ . This completes the verification of the claim.

Thus, if Theorem 2 is false, there are infinitely many primes p for which $A[p]$ contains a G -submodule isomorphic to either $\mathbf{Z}/p\mathbf{Z}$ or to μ_p . Of course, the former case can occur only a finite number of times, since $A(k)$ is finite. One way to rule out the latter case is to argue that whenever μ_p is a submodule of $A[p]$, the group $\mathbf{Z}/p\mathbf{Z}$ is a quotient of the dual of $A[p]$, which is the kernel of multiplication by p on the abelian variety A^\vee dual to A . In other words, if μ_p occurs as a submodule of $A[p]$, then there is an abelian variety isogenous to A^\vee (and therefore in fact to A) which has a rational point of order p over k . Therefore p is a divisor of the order of a finite group that may be specified in advance, viz. the group of rational points of any reduction of A at a good unramified prime of k of residue characteristic different from 2. (See the appendix to Katz's recent paper [2] for a discussion of this point.)

Proof of Theorem 3. Suppose that p is a prime such that V_p^H is non-zero. We again choose W to be an irreducible G -submodule (i.e., $\mathbf{Q}_p[G]$ -submodule) of V_p^H . Because the action of G on W is abelian, and because W is simple, each element of G acts semisimply on W . Since A/k is semistable, it follows that the homomorphism

$$\rho: G \rightarrow \text{Aut}(W)$$

giving the action of G on W is unramified at all primes of k not dividing p . Therefore, ρ factors through $\text{Gal}(K_p/k)$ in view of the lemma and the subsequent replacement $k \rightarrow k'$. In other words, starting from the hypothesis that the p -torsion subgroup of $A(K)$ is infinite, we have deduced that the p -torsion subgroup of $A(K_p)$ is infinite.

Of course, this situation is ruled out by the theorem of Imai and Serre mentioned above. Nevertheless, we will sketch for the reader's convenience an argument which leads to a contradiction. Let v be a place of k dividing p , and let $D \subset G$ be a decomposition group for v . By ([SGA 71], IX, Prop. 5.6), the D -module V_p is an extension of D -modules attached to p -divisible groups over the integer ring of the completion of k at v . Because of Tate's theory [7], the semisimplification V_p^{ss} of the D -module V_p has a Hodge-Tate decomposition. (Here we should remark that submodules and quotients of Hodge-Tate modules are again Hodge-Tate.) Since W is semisimple as a D -module (because semisimple and *abelian* as a G -module), W may be viewed as a submodule of V_p^{ss} . Therefore, W is a Hodge-Tate module.

By ([6], III, Appendix), we know that ρ is a locally algebraic abelian representation of G . Using this information, plus the fact that ρ factors through $\text{Gal}(K_p/k)$, we find that there is an open subgroup G_0 of G with the following property: the restriction of ρ to G_0 is the direct sum of 1-dimensional representations, each described by an integral power χ_p^n of the standard cyclotomic character $\chi_p: G \rightarrow \mathbf{Z}_p^*$. After replacing k by a finite extension, we may assume that G_0 is G . Take a prime w of k which is prime to p and such that A has good reduction at w . Let $g \in G$ be a Frobenius element for w . The eigenvalues of $\rho(g)$ will be integral powers of $\chi_p(g)$, i.e., of the norm Nw of w . However, by a well known theorem of Weil, these eigenvalues all have archimedian absolute values equal to $(Nw)^{1/2}$. This contradiction completes the proof of Theorem 3.

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