Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 27 (1981)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: DESCARTES, EULER, POINCARÉ, PÓLYA—AND POLYHEDRA

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Kapitel: 3. Historical comment and summary

DOI: https://doi.org/10.5169/seals-51756

of P_3 of dimension [1]) $(n-1) - k$. Moreover, the incidence relations are carried over by this duality; thus if, in P_2, every $(i-1)$-face is incident with [2]) s_i i-faces, then, in P_3, every $(n-i)$-face is incident with s_i $(n-i-1)$-faces (and there is a symmetrical statement interchanging P_2 and P_3). In this sense P_1 is selfdual. Figure 8 displays these dualities for $n = 4$, as well as the value of Δ.

3. Historical comment and summary

René Descartes (1596–1650) and Leonhard Euler (1707–1783) worked on these subjects independently—yet, as we have seen, Pólya (1887–) has shown that their seemingly different formulae for convex polyhedra homeomorphic to S^2 are entirely equivalent to each other. One might believe from the evidence that Euler may have known about Descartes' work on this subject. That would be an erroneous assumption since Descartes' work on this matter [5] was not printed until a century after Euler's death (see [1], p. 56).

Euler [6] offered a variety of verifications but no formal proof of his formula. We have observed that each of the formulae is somewhat surprising by itself and that their connection rather defies intuition since at first glance they seem to be dealing with different qualitative aspects of polyhedra. As a matter of fact neither Euler's nor Descartes' formula is easy to *prove* independently; yet, as we have seen, it is not at all difficult to follow Pólya's proof that the two formulae are equivalent.

The formulae diverge in higher dimensions so that their relationship is a special phenomenon of dimension 2. Euler's formula was generalized by Ludwig Schläfli [9], a Swiss mathematician of the 19th century (1814–1895), who described, in effect, the Euler-Poincaré characteristic of an n-dimensional sphere S^n, subdivided as a *polytope*, a combinatorial structure attributed by Coxeter to Reinhold Hoppe [11]. Poincaré (1854-1912) gave a definition of the Euler-Poincaré characteristic for arbitrary polyhedra, and one proves now, by invoking the topological invariance of the homology groups (see [12]) that the Euler-Poincaré characteristic is a topological invariant.

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[1]) The precise form of this duality shows how "correct" it is to regard S^{n-1} as $(n-1)$-dimensional, rather than n-dimensional.

[2]) In fact, $s_i = 2(n-i-1)$.

On the other hand, there will be no straightforward generalization to higher dimensions of Descartes' formula for the total angular defect of a polyhedron homeomorphic to S^2, since this defect ceases in higher dimensions to be a topological invariant. However it remains, under suitable restrictions on the cellular structure, a combinatorial invariant in a certain strict sense and thus independent of the underlying geometry of the polyhedron.

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(Reçu le 25 mars 1981)

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