Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SCHUBERT CALCULUS OF A COXETER GROUP
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Kapitel: 1. Coxeter groups
DOI: https://doi.org/10.5169/seals-51740

1. Coxeter groups

We begin by reviewing some of the elementary theory of Coxeter groups. Some detail is included to avoid any oblique use of the crystallographic condition. Following Bourbaki [6, IV] we say (W, S) is a *finite Coxeter system* if W is a finite group given by the presentation $\langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where m_{ij} is the order of $s_i s_j$. It is possible [6, V] to construct a real Euclidean space V and a root system (Δ, Σ) in V that "geometrically realizes" (W, S). By this we mean the following. If $\gamma \in \Delta$ then

$$s_\gamma(x) = x - (x, \gamma^v)\gamma \qquad \left(\text{co-root} \quad \gamma^v = \frac{2\gamma}{(\gamma, \gamma)}\right)$$

is the reflection through the hyperplane perpendicular to the root γ, and we can form the subgroup $W(\Delta)$ of $GL(V)$ generated by the s_γ's, $\gamma \in \Delta$. In fact, the s_α's, $\alpha \in \Sigma$, generate $W(\Delta)$ and we call the pair $(W(\Delta), \{s_\alpha : \alpha \in \Sigma\})$ the *Weyl system* of (Δ, Σ). Coxeter [9] proved that the Weyl system is always a Coxeter system and if this pair is isomorphic (in the obvious sense) to (W, S) we say (Δ, Σ) is a geometric realization of (W, S). Of course, the choice of such a (Δ, Σ) is not unique. But clearly up to a rigid motion of V, the root system is determined by the lengths of the simple roots.

If the lengths can be chosen so that $(\alpha, \beta^v) \in \mathbf{Z}$ for all $\alpha, \beta \in \Sigma$, we say W is *crystallographic* (or a Weyl group). Geometrically, this means that the $\mathbf{Z}$-lattice generated by Σ is preserved by W. As mentioned in the introduction, even this choice of relative lengths is not necessarily unique.

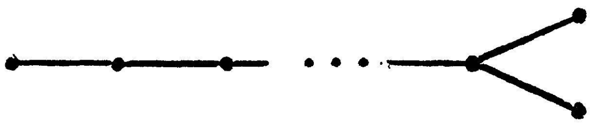
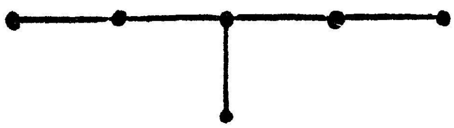
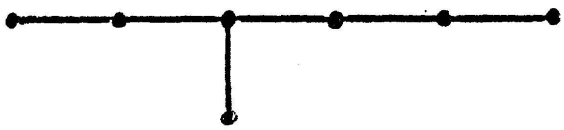
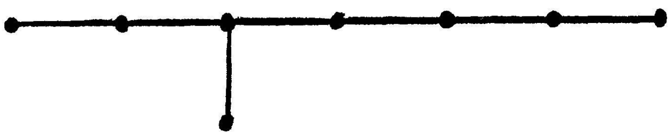
We can choose a vector $t \in V$, such that $(t, \alpha) > 0$ for all $\alpha \in \Sigma$ (i.e. t is in the fundamental chamber C). This vector decomposes the roots $\Delta = \Delta^+ \coprod \Delta^-$ where

$$\Delta^+ = \{\gamma \in \Delta : (\gamma, t) > 0\}$$

and $\Delta^- = -\Delta^+$. Note that $|\Delta^+| = N = \frac{1}{2}|\Delta|$, where N is the number of reflections in W as described in the introduction.

It is now customary to attach an edge labelled graph to (W, S) called the Coxeter graph. The nodes correspond to the elements of S and s_i is attached to s_j by an edge if $m_{ij} \geqslant 3$, and if also $m_{ij} > 3$ the edge is labelled with the number m_{ij}. In 1934, Coxeter [9] classified the Coxeter groups with connected graphs and showed that every Coxeter group is a product of the "connected" components. The classification of the irreducible Coxeter groups along with the fundamental degrees is

TABLE

W	Coxeter graph	$d_1, \ldots, d_n$
A_n		2, 3, ..., $n+1$
B_n		2, 4, ..., $2n$
D_n		2, 4, ..., $2(n-2)$, $2(n-1)$, n
E_6		2, 5, 6, 8, 9, 12
E_7		2, 6, 8, 10, 12, 14, 18
E_8		2, 8, 12, 14, 18, 20, 24, 30
F_4		2, 6, 8, 12
G_2		2, 6
H_3		2, 6, 10
H_4		2, 12, 20, 30
$I_2(m)$		2, m

We will assume throughout that W is irreducible.

The crystallographic Coxeter groups and their root systems are well-known and correspond up to a choice of relative lengths of the simple roots to the Cartan classification of simple Lie algebras over the complex numbers. The dihedral groups are the Weyl groups of $I_2(m)$, and are the symmetry groups of a regular m-gon (from which it is easy to construct (Δ, Σ)). The group H_3 is isomorphic to a product of $\mathbf{Z}_2$ and an alternating group on five letters and H_4 is the symmetry group of a certain 4-dimensional polytope [9, 10].

The primary piece of structure available on a Coxeter group is the *length function* $l : W \to \mathbf{N}$, where $l(w)$ is defined as the minimal length of an expression of w in the generators S. If $l(w) = k$ and $w = s_1 \ldots s_k$, $s_i \in S$, we call this a *reduced decomposition* of W. There is an alternative intrinsic description.

LEMMA 1.1. *Let* Γ_w *denote the set of* $\gamma \in \Delta^+$ *such that* $w(\gamma) \in \Delta^-$, *then*

(i) $|\Gamma_{ws_\alpha}| = |\Gamma_w| \pm 1$ *if and only if* $w(\alpha) \in \Delta^{\pm}$,

(ii) $l(w) = |\Gamma_w|$,

(iii) $l(ws_\alpha) = l(w) \pm 1$ *if and only if* $w(\alpha) \in \Delta^{\pm}$.

Proof. To see (i) one need only recall that $\Gamma_{s_\alpha} = \{\alpha\}$. This first assertion then implies $|\Gamma_w| \leqslant l(w)$. The other inequality follows from an induction on $|\Gamma_w|$ and then (ii) follows. (iii) is immediate from (i) and (ii).

The next piece of structure on the Coxeter group we require is the so-called Bruhat ordering [13]. We define $w' \to w$ (intuitively, w' is an immediate predecessor of w if there exists a positive root γ such that $\sigma_\gamma w = w'$ and $l(w') = l(w) + 1$. (We will often adorn $\to$ with the unique such γ.) Since W is transitive on the roots and $ws_\alpha w^{-1} = s_{w(\alpha)}$ the first condition is equivalent to $w' w^{-1}$ being a conjugate of a fundamental reflection $s \in S$. The *Bruhat order* $<$ on W is the transitive closure of the ordering $\to$. Note that l is forced to be strictly order-preserving so that the two pieces of structure we have introduced are compatible. We can now relate $\to$ to any particular reduced decomposition of w.

LEMMA 1.2. *If* $w = s_1 \ldots s_k$ *is a reduced decomposition, then* $w' \to w$ *only if* $w' = w_{\hat{i}}$ *where* $w_{\hat{i}} = s_1 \ldots \hat{s}_i \ldots s_k$ *(and* $\hat{}$ *denotes deletion).*

Proof. See Theorem 1.1 (III) in [13].

Hence, in general, the Bruhat ordering corresponds to the subwords of any reduced decomposition. So, for any i we can find a $\gamma \in \Delta^+$ such that $s_\gamma w_{\hat{i}} = w$. The next result describes these roots γ both specifically and abstractly.

LEMMA 1.3. *If* $w = s_1 \dots s_k$ *is a reduced decomposition, define* $\theta_i = s_1 \dots s_{i-1}(\alpha_i)$ *where* $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$. *Then the following sets are equal*

(i) $\Gamma_{w-1} = \Delta^+ \cap w(\Delta^-)$,

(ii) $\{\theta_i\}_{1 \leq i \leq k}$,

(iii) $\{\gamma \in \Delta^+ : s_\gamma w_i = w\}$.

Proof. (i) $\subseteq$ (ii). Let $\gamma \in \Delta^+$ and $w^{-1}(\gamma) \in \Delta^-$. Let j be the smallest number such that $s_j \dots s_1(\gamma) \in \Delta^-$. Then $\alpha_j = s_{j-1} \dots s_1(\gamma)$. Hence $\gamma = \theta_j$.

(ii) $\subseteq$ (iii). It suffices to compute

$$\begin{aligned} s_{\theta_i} \hat{w}_i &= s_{s_1 \dots s_{i-t}(\alpha_i)}(s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_{i-l} s_i s_{i-l} \dots s_1 (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_k = w . \end{aligned}$$

But now $|\Gamma_{w-1}| = l(w^{-1}) = l(w) = k$, by (1.1) and certainly $|\{\gamma \in \Delta^+ : s_\gamma w_{\hat{i}} = w\}| \leqslant k$, so all three sets must be equal.

Remark. Though the θ_i's are defined in terms of a reduced decomposition, (1.3 i) shows that they are actually independent of the choice made.

We now recall that the Bruhat order on W possesses a unique top element of greatest length.

LEMMA 1.4. *There exist a unique element* $w_0 \in W$ *such that* $l(w_0) = N$. *In addition,* $w_0 \geqslant w$, *for all* $w \in W$, $w_0^2 = 1$ *and* $l(ww_0) = l(w_0) - l(w)$.

Proof. One knows that W acts simply transitively on the chambers and w_0 is chosen to be the unique element satisfying $w_0 C = -C$. The rest is standard, see [6, p. 43].

Finally, we make some remarks on the (anti) invariant theory of Coxeter groups. The main result is

PROPOSITION 1.5. *If* (W, S) *is a Coxeter system, then the invariant algebra* $S(V)^W$ *has* $|S|$ *algebraically independent generators of degrees* $2 = d_1, d_2, \dots, d_n$. *Equivalently,* $S(V)$ *is a free* $S(V)^W$*-module.*

Proof. This follows immediately from Chevalley's theorem [8].

Remark. It is often useful in this context to think of W as the Galois group of the rational function field $\overline{S(V)}$ over the rational function field $\overline{S(V)^W}$ of the invariants. We exploit this point of view in the next section.

There is also a theory of anti-invariants, i.e. polynomials $u \in S(V)$ such that $w \cdot u = (-1)^{l(w)} u$. The algebra of anti-invariants is written $S(V)^{-W}$. It is a free module of rank 1 over $S(V)^W$ generated by the element $d = \prod_{\gamma \in \Delta +} \gamma \in S_N(V)$. The corresponding "anti-averaging" operating is

$$\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u.$$

2. Demazure's basis theorem

Let $\varepsilon : S(V) \to S_0(V) \approx \mathbf{R}$ denote the projection map. We begin by defining certain operators on $S(V)$, whose composition with ε should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let (W, S) be a Coxeter system and (Δ, Σ) a geometric realization of it.

Definition 2.1. If $\alpha \in \Delta$, define $\Delta_\alpha = \alpha^{-1}(1 - s_\alpha)$ as an $S(V)^W$-endomorphism of $S(V)$. (Note the division is legitimate since s_α is the identity on the $\ker(\alpha) = \alpha^\perp$; thinking of α as a linear form $x \mapsto (x, \alpha)$ in $V^* = S_1(V)$, of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

Lemma 2.2. *If* $w \in W, \alpha \in \Delta, u, v \in S(V)$ *then*

(i) $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)}$,

(ii) $\Delta_\alpha^2 = 0$,

(iii) $s_\alpha = 1 - \alpha \Delta_\alpha$,

(iv) $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$ *(where the superscript denotes invariants)*

(v) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$,

(vi) $\Delta_\alpha(I_W) \subset I_W$,

(vii) $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^v, \omega) s_\alpha$,

where ω^* *denotes the operator multiplication by* ω.

We now define $\mathbb{A}_W$ to be the subalgebra of the algebra of endomorphisms $\mathrm{End}(S(V))$ generated by the Δ_α's ($\alpha \in \Delta$) and ω^*, $\omega \in S(V)$. Note Δ_α decreases the grading by (-1) and $W \subseteq \mathbb{A}_W$ by (2.2 iii).