5. H_w AS A AND PARABOLICS

Example. Recall that in H_{Σ_3} we computed

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) .$$

By (4.6), one can compute

$$\begin{aligned} X_{s_\alpha}^2 &= X_{s_\beta s_\alpha} \\ X_{s_\beta} X_{s_\alpha} &= X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta} \\ X_{s_\beta}^2 &= X_{s_\alpha s_\beta} \end{aligned}$$

and this confirms our earlier computation.

5. H_W AS A W-MODULE AND PARABOLICS

If (W, S) is a Coxeter system and $\theta \subseteq S$ then (W_θ, θ) is also a Coxeter system [6, p. 20] and W_θ is called a *parabolic* subgroup of W. In addition, it is easy to see that a geometric realization (Δ, Σ) of (W, S) can be restricted to a geometric realization of (W_θ, θ). The collection $\{W_\theta\}_{\theta \subset S}$ of parabolic forms a lattice of $2^{|S|}$ distinct subgroups where, for example, $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$. We will eventually be concerned with the set of left cosets of W_θ in W. We define $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$. The following basic result is well-known [6, p. 37 and p. 45].

THEOREM 5.1. *Every element w of W can be uniquely expressed as $w^\theta \cdot w_\theta$ with $w^\theta \in W^\theta$, $w_\theta \in W_\theta$ and furthermore $l(w) = l(w^\theta) + l(w_\theta)$.*

This immediately yields

COROLLARY 5.2. *W^θ is a complete set of left coset representations for W_θ in W and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra $H_W^{W_\theta}$ of W_θ-invariants in H_W. The most straightforward approach is to compute exactly the action of W on H_W. This is easily done by exploiting the computation (4.1).

THEOREM 5.3. *The structure of H_W as a W-module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \displaystyle\sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$

Proof. As in (4.5), choose A such that $\varepsilon\,\Delta_g(A) = \delta_{gw}$. Then, since c is a W-map

$$\begin{aligned}
s_\alpha X_w &= c(s_\alpha A) = \sum_{w' \in W} \varepsilon\,\Delta_{w'}(s_\alpha A)\, X_{w'} \\
&= \sum_{w' \in W} \varepsilon\,\Delta_{w'}(1-\alpha^*\Delta_\alpha)(A)\, X_{w'} \\
&= X_w - \sum_{w' \neq w} (\varepsilon\Delta_{w'}\alpha^*)\,\Delta_\alpha(A)\, X_{w'} \\
&= X_w - \sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \alpha)\,\varepsilon\,\Delta_{gs_\alpha}(A)\, X_{w'} \qquad \text{by (4.1)} \\
&= \sum_{\substack{g \xrightarrow{\gamma} w' \\ l(gs_\alpha) = l(g)+1 \\ gs_\alpha = w}} (g^{-1}(\gamma)^v, \alpha)\, X_{w'} \\
&= X_w - \sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^v, \alpha)\, X_{w'}
\end{aligned}$$

Note, that the summation in the next to the last line is non-vacuous if and only if $l(ws_\alpha) = l(w) - 1$. This completes the proof.

COROLLARY 5.4. *$X_w \in H_W^{W_\theta}$ if $w \in W^\theta$.*

Proof. Immediate from (5.3) and the definition of W^θ.

The following elementary result shows that the X_w, $w \in W^\theta$, are actually an $\mathbf{R}$-basis for $H_W^{W_\theta}$.

LEMMA. *If a finite group G acts on a real vector space V via the regular representation and H is a subgroup of G, then*

$$\dim_{\mathbf{R}}(V^H) = [G:H].$$

Proof. Let $\{e_g\}_{gG}$ be a basis for V, so that

$$g' \cdot e_g = e_{gg'}$$

Then if $\chi = \sum_{g\in G} \xi_g \in V^H$, we claim $\xi_g = \xi_{g'}$, if $g \equiv g' \pmod H$. Indeed, if $g = g'h$, $h \in H$, then

$$\begin{aligned}
\xi_{g'} &= \text{coefficient of } e_{g'}\ \chi \text{ in} \\
&= \text{coefficient of } e_{g'} \text{ in } h^{-1}\chi\cdot \\
&= \text{coefficient of } e_{g'h} \text{ in } \chi \\
&= \xi_g.
\end{aligned}$$

Hence, there are at most $[G : H]$ free parameters in determining $\chi \in V^H$ and clearly each choice gives an invariant. This finishes the proof.

COROLLARY 5.6. *dim* $(H_W^{W_\theta}) = [W : W_\theta] = |W^\theta|$ *and the* X_w, $w \in W^\theta$, *are an* **R**-*basis for* $H_W^{W_\theta}$.

Proof. Chevalley [8] has shown that S_W, hence H_W, is abstractly equivalent to the regular representation of W, as a W-module. Hence, (5.5) applies and the result follows.

It is now possible to "restrict" the Pieri formula (4.5) for H_W to $H_W^{W_\theta}$. We have

THEOREM 5.7. *If* $w, w' \in W^\theta$ *and in* H_W

$$X_w \cdot X_{w'} = \sum_{w'' \in W} c(w, w', w'') X_{w''}$$

then in $H_W^{W_\theta}$

$$X_w \cdot X_{w'} = \sum_{w'' \in W^\theta} c(w, w', w'') X_{w''}$$

Proof. One need only observe that the vector space map $r : H_W \to H_W^{W_\theta}$ given by

$$r(X_w) = \begin{cases} X_w & \text{if } w \in W^\theta \\ 0 & \text{otherwise} \end{cases}$$

is a retraction. Then, applying r to both sides of the first equation yields the second equation since the invariants form a subalgebra.

This result will be useful in the next section for computing inside the algebra of W_θ-invariants. Notice that an appropriate Giambelli formula for $H_W^{W_\theta}$ is not as easily obtained. This is because the Giambelli formula for H_W gives X_w as a polynomial in the X_{s_α}'s and not all of these are in the algebra $H_W^{W_\theta}$, so this is not quite the right thing.

6. APPLICATION: THE COMBINATORICS OF THE CLASSICAL PIERI FORMULA

In the last section we saw that given a pair (W, W_θ) of a Coxeter group and a parabolic subgroup, one could construct a formula to describe the multiplication of Schubert generators in the invariant subalgebra $H_W^{W_\theta}$. In this section, we examine the particular case $(\Sigma_{n+k}, \Sigma_k \times \Sigma_n)$ where Σ_m denotes the symmetric group on m letters. Indeed, Σ_{n+k} is the Weyl group of the root system of type A_{n+k-1}, which we recall briefly here. Let $V' = \mathbf{R}^{n+k}$